

## Improved Version of Hermite-Hadamard Inequalities by Generalized Fuzzy Fractional Operators in Aspects of Convexity

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### ABSTRACT

Special functions play a crucial role in the development of fractional calculus and have extensive applications in solving problems involving differential equations, integrals, and series solutions. In this thesis, we introduce two-dimensional Riemann-Liouville fractional operators, Riemann-Liouville fuzzy fractional operators (FFOs), and generalized fuzzy fractional operators incorporating the extended Bessel-Maitland function ( $E^1\nu\text{BMF}$ ) as a kernel. We also construct two-dimensional  $h$ -Godunova-Levin convex functions, coordinated pre-invex functions, and various types of fuzzy convex and non-convex functions. Furthermore, we derive new forms of Hermite-Hadamard (H-H) and trapezoid-type fractional integral inequalities for these newly defined two-dimensional convex and pre-invex functions. We also examine the existence and generalization of H-H and trapezoid-type inequalities using the framework of fuzzy interval-valued functions (FIVFs), which extend classical interval- and real-valued functions. By effectively applying generalized fuzzy fractional operators with ( $E^1\nu\text{BMF}$ ) kernels, we establish new inequalities and discuss additional applications arising from our main results, presented in the form of corollaries.

**Keywords:** Riemann–Liouville fractional operators; Fuzzy fractional operators

(FFOs); Extended Bessel–Maitland function (E1vBMF);  $h$ -Godunova–Levin convex functions; Pre-invex functions; Fuzzy interval-valued functions (FIVFs).

## Introduction

Special functions occupy a central role in pure, applied, and computational mathematics due to their extensive utility in solving series, integral equations, and complex analytical problems. Functions such as the beta, gamma, hypergeometric, and trigonometric functions are foundational tools that facilitate mathematical modeling across various domains. Historically, many of these functions were formalized during the nineteenth century, primarily by Swiss mathematicians, whose contributions laid the groundwork for much of modern mathematical analysis. The generating function approach, along with associated differential and integral equations, further advanced the study of these functions, leading to significant applications in functional analysis, numerical analysis, and fractional analysis [1–4]. The advent of fractional calculus has introduced new perspectives on differential and integral operations by extending their order from integers to real or complex numbers. This generalized framework allows for the modeling of phenomena that cannot be adequately described by traditional calculus. The historical roots of fractional calculus trace back nearly three centuries, originating with a question posed by G. W. Leibniz and Marquis de l'Hospital in 1695 regarding the concept of a semi-derivative. This inquiry stimulated interest among renowned mathematicians including Euler, Liouville, Riemann, Laplace, Grünwald, and Letnikov, who collectively contributed to the formal development of fractional operators [5–14].

The Riemann–Liouville fractional operator, in particular, has played a pivotal role in the generalization of classical integral inequalities into fractional forms, thereby expanding the theoretical and applied horizons of mathematical analysis [8–10]. Within this context, fractional integral inequalities have become a vital area of study, especially in approximation theory and mathematical analysis. Among these, the Hermite–Hadamard inequality, first introduced by Charles Hermite and Jacques Hadamard, represents a fundamental result for convex functions defined on real intervals. The concept of convexity, integral to optimization, control theory, economics, and game theory, continues to attract significant research attention [15–17]. The intersection of convexity and integral inequalities has inspired numerous

studies exploring generalizations such as pre-invexity and related concepts, including their two-dimensional extensions [18–20].

A recent direction of research focuses on the integration of fuzzy interval-valued functions with fractional calculus. Introduced by Ramon E. Moore in 1966, interval analysis provides a robust mathematical framework for uncertainty quantification and optimization. Its fusion with fuzzy set theory has led to the development of fuzzy fractional integral inequalities, enabling the modeling of systems characterized by uncertainty or imprecise information [21–23]. These developments have significant implications for global optimization and constraint-solving algorithms, further emphasizing the interdisciplinary relevance of fractional inequalities. Another crucial component of this research is the Bessel function, originally proposed in 1732 by Daniel Bernoulli to describe oscillatory motion. Later developments by Euler, Friedrich Wilhelm Bessel, and other mathematicians broadened its theoretical foundation and application spectrum. Bessel functions now play an indispensable role in various branches of applied mathematics and physics, including electromagnetism, diffraction theory, and hydrodynamics [24–27]. The generalization of Bessel functions has produced novel classes such as the Bessel–Maitland function, introduced by Edward Maitland Wright, which extends the classical Bessel function through series representations, yielding new analytical and computational results [9, 28, 29]. The present study builds upon these theoretical foundations by exploring fractional integral inequalities involving fuzzy interval-valued functions and special kernels, such as the exponential Bessel–Maitland function (E1vBMF). This work is structured into five chapters. The first chapter provides the necessary preliminary definitions and mathematical background. The second chapter modifies the Hermite–Hadamard and trapezoidal-type inequalities in two dimensions using the Riemann–Liouville fractional integral operator. The third chapter extends these inequalities into the fuzzy fractional domain. The fourth chapter investigates refinements of these inequalities through the application of the E1vBMF function as a kernel, analyzing their behavior in fuzzy environments. Finally, the fifth chapter presents the conclusions and outlines potential directions for future research. Through this investigation, the study contributes to the ongoing advancement of fractional analysis, convexity theory, and fuzzy interval mathematics

by establishing new inequalities and frameworks that can be applied in both theoretical and applied mathematical contexts.

## Preliminaries

In this we discuss fundamental definitions related to special functions, fractional operators, and their generalizations. These concepts serve as the foundation for deriving our main results.

|                                                    |                                                                                                                                                                                                                                                                  |                                                                                                                                                                                                                                                                                                                                                                                                             |
|----------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <b>Gamma Function:</b>                             | <b>The gamma function [39] is defined for <math>\Re(z) &gt; 0</math> via the integral representation:</b>                                                                                                                                                        | $\Gamma(z) = \int_0^\infty g^{z-1} e^{-g} dg.$                                                                                                                                                                                                                                                                                                                                                              |
| <b>Beta function</b>                               | Beta function defined for $\Re(x) > 0$ , $\Re(z) > 0$ in integral representation and also with gamma functions relations can be write as:                                                                                                                        | $B(x, z) = \int_0^1 g^{x-1} (1-g)^{z-1} dg = \frac{\Gamma(x)\Gamma(z)}{\Gamma(x+z)}$                                                                                                                                                                                                                                                                                                                        |
| <b>The classical Gauss hypergeometric function</b> | The Gauss hypergeometric function, as defined by Beukers [39], is valid for $ z  < 1$ with parameters $\xi, \zeta, \delta \in \mathbb{C}$ and $\delta \neq 0$ , where $(\xi)_n$ , $(\zeta)_n$ , and $(\delta)_n$ denote the Pochhammer symbols, and is given by: | ${}_2F_1(\xi, \zeta, \delta, z) = \sum_{n=0}^{\infty} \frac{(\xi)_n (\zeta)_n z^n}{(\delta)_n n!}.$ <p>When <math>z=1</math> then the Gauss hypergeometric function in the form of gamma function can be defined for <math>\Re(\delta - \xi - \zeta) &gt; 0</math> as</p> ${}_2F_1(\xi, \zeta, \delta, 1) = \frac{\Gamma(\delta)\Gamma(\delta - \xi - \zeta)}{\Gamma(\delta - \xi)\Gamma(\delta - \zeta)}.$ |
| <b>Dirichlet formula</b>                           | Dirichlet formula (Fubini's theorem), [24] defined as                                                                                                                                                                                                            | $\int_x^y du \int_x^u z(u, v) dv = \int_x^y dv \int_v^y z(u, v) du.$                                                                                                                                                                                                                                                                                                                                        |
| <b>Lebesgue measurable space</b>                   | The Lebesgue measurable space [40] is defined for $c > a$ as follows:                                                                                                                                                                                            | $\mathbf{L}(a, c) = \{g(z) :$ $\frac{\gamma(x+a)}{\gamma(x+b)} = x^{a-b} [1 + \frac{(a-b)(a+b-1)}{2x} + o(\frac{1}{x^2})].$                                                                                                                                                                                                                                                                                 |
| <b>Bessel-Maitland</b>                             | Edward Maitland Wright [41] introduced a function known as the Fox-Wright                                                                                                                                                                                        | $J_{\beta}^{\alpha}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(\alpha n + \beta + 1)} = \phi(\alpha, \beta + 1; -z),$                                                                                                                                                                                                                                                                                 |

|                                                     |                                                                                                                                                                                                                                                                                                                                               |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |
|-----------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <b>function</b>                                     | function,<br>which generalizes the Bessel function. The Bessel-Maitland function is represented as:                                                                                                                                                                                                                                           |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |
| <b>Generalization of Bessel-Maitland function</b>   | Generalization of Bessel-Maitland function in four parameters defined by [21] for $\Re(\alpha) > 0$ , $\Re(\beta) \geq -1$ , $\Re(\gamma) > 0$ and $q \in (0,1) \cup \mathbb{N}$ as                                                                                                                                                           | $J_{\beta,q}^{\alpha,\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} (-z)^n}{n! \Gamma(\alpha n + \beta + 1)},$ <p>where <math>\alpha, \beta, \gamma \in \mathbb{C}</math>.</p>                                                                                                                                                                                                                                                                                                             |
| <b>An extension of the Bessel-Maitland function</b> | The extended Bessel-Maitland function, which serves as a further generalization of the Bessel-Maitland function [22], is defined for $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\Re(\alpha) > 0, \Re(\beta) \geq -1, \Re(\gamma) > 0, \Re(\delta) > 0$ . Also, let $b, d \in \mathbb{R}^+$ with $b < \Re(\alpha) + d$ represents as | $J_{\beta,\gamma,\delta}^{\alpha,b,d}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{bn} (-z)^n}{\Gamma(\alpha n + \beta + 1) (\delta)_{dn}}.$ <p>Throughout this work, we use the notation <math>\mathbb{C}</math> for the set of complex numbers, <math>\mathbb{R}</math> for the set of positive real numbers, and <math>\mathbb{N}</math> for the set of positive integers. The function introduced herein is a generalized form of the Bessel-Maitland function, defined with six parameters.</p> |
| <b>Generalized Bessel-Maitland function-III</b>     | The generalized Bessel-Maitland function [42], is elucidate as follows                                                                                                                                                                                                                                                                        | $\mathbb{J}_{\nu,\eta_1,\rho_1,\gamma_1}^{\mu,\xi,m,\sigma}(s) = \sum_{n=0}^{\infty} \frac{(\eta_1)_{\xi n} (\gamma_1)_{\sigma n} (-s)^n}{\Gamma(\beta n + \alpha + 1) (\rho_1)_{mn}}.$                                                                                                                                                                                                                                                                                                          |

Following the above-mentioned equations, our study also incorporated an in-depth focus on several advanced mathematical constructs and operators. These include the Bessel-Maitland function-IV, Generalized Srivastava fractional integral operator, Ghayasuddin fractional integral operator, Extension of generalized Ghayasuddin fractional integral operator, New version of generalized Ghayasuddin fractional integral operator, Bessel-Maitland k-function, Generalized Bessel-Maitland k-function, Extended Bessel-Maitland k-function, Srivastava fractional integral k-

operator, Generalized Srivastava fractional integral k-operator, Ghayasuddin fractional integral k-operator ensuring a comprehensive treatment of both classical and k-generalized formulations within the fractional calculus framework.

## Analysis of Fractional Double Integral Inequalities and Their Consequences

In this section, we explore two-dimensional h-Godunova-Levin convex functions and coordinated pre-invexity. We examine Hermite-Hadamard and trapezoid-type integral inequalities by effectively applying two-dimensional fractional integral operators. This approach allows us to derive generalized fractional inequalities that extend beyond the one-dimensional case, marking significant advancements in the field of analysis.

### 3.1 Fractional Hermite-Hadamard Integral Inequalities for two-Dimensional h-Godunova-Levin Convex Functions

Here, we investigate the Hermite-Hadamard type fractional integral inequalities for two dimensional h-Godunova-Levin convex function by using two-dimension fractional integral operators, and discuss its analysis in the form of corollaries, which present the Hermite-Hadamard fractional integral inequalities in aspects of two dimension h- Godunova-Levin convex function.

**Theorem 3.1.1.** Let  $h : (0, 1) \rightarrow \mathbb{R}$  be positive function,  $h(\theta) \neq 0$ , and  $\varsigma : [\vartheta, \wp] \times [k_3, k_4] \rightarrow \mathbb{R}$  be two dimension h-Godunova-Levin function for  $0 < \vartheta < \wp, 0 < k_3 < k_4$ ,  $\varsigma \in L_1[\vartheta, \wp], L_1^*[k_3, k_4]$ , then the following inequality holds

$$\begin{aligned} \frac{h(\frac{1}{2})}{2} \varsigma \left[ \frac{\vartheta + \wp}{2}, \frac{k_3 + k_4}{2} \right] &\leq \frac{\gamma(\eta + 1)}{2(\wp - \vartheta)^\eta (k_4 - k_3)^\eta} \left[ \Xi_{(\vartheta, k_3)+}^\eta \varsigma(\wp, k_4) + \Xi_{(\wp, k_4)-}^\eta \varsigma(\vartheta, k_3) \right] \\ &\leq \eta \frac{\varsigma \left[ \frac{\vartheta + \wp, k_3 + k_4}{2} \right]}{2} \int_0^1 \int_0^1 \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \theta^{\eta-1} r^{\eta-1} d\theta dr \end{aligned}$$

*Proof.* Consider the definition of two dimension h-Godunova-Levin convex function

$$\varsigma[\theta x + (1-\theta)z, \theta y + (1-\theta)w] \leq \frac{\varsigma(x, y)}{h(\theta)} + \frac{\varsigma(z, w)}{h(1-\theta)}. \quad (3.1)$$

Putting the values  $x = \theta\vartheta + (1-\theta)\wp, y = rk_3 + (1-r)k_4, z = (1-\theta)\vartheta + \theta\wp, w =$



$(1-r)k_3 + rk_4$  and  $\delta = \frac{1}{2}$  in equation (3.1), we have

$$h\left(\frac{1}{2}\right)\zeta\left[\frac{\bar{\theta}+\rho}{2}, \frac{k_3+k_4}{2}\right] \leq \zeta[\theta\bar{\theta} + (1-\theta)\rho, rk_3 + (1-r)k_4] + \zeta[(1-\theta)\bar{\theta} + \theta\rho, (1-r)k_3 + rk_4]. \quad (3.2)$$

After multiplication both sides of the equation (3.2) by  $\theta^{\eta-1}r^{\eta-1}$  and then integrate with respect to  $\theta$ , and  $r$  over the interval  $[0, 1]$ , we obtain

$$h\left(\frac{1}{2}\right)\zeta\left[\frac{\bar{\theta}+\rho}{2}, \frac{k_3+k_4}{2}\right] \int_0^1 \int_0^1 \theta^{\eta-1}r^{\eta-1}d\theta dr \leq \int_0^1 \int_0^1 \zeta[\theta\bar{\theta} + (1-\theta)\rho, rk_3 + (1-r)k_4] \theta^{\eta-1}r^{\eta-1}d\theta dr + \int_0^1 \int_0^1 \zeta[(1-\theta)\bar{\theta} + \theta\rho, (1-r)k_3 + rk_4] \theta^{\eta-1}r^{\eta-1}d\theta dr \quad (3.3)$$

By putting the values  $\theta\bar{\theta} + (1-\theta)\rho = t, rk_3 + (1-r)k_4 = s, (1-\theta)\bar{\theta} + \theta\rho = v, (1-r)k_3 + rk_4 = w$  in the equation (3.3), we obtain

$$h\left(\frac{1}{2}\right)\zeta\left[\frac{\bar{\theta}+\rho}{2}, \frac{k_3+k_4}{2}\right] \frac{1}{\eta^2} \leq \int_{\rho}^{\bar{\theta}} \int_{k_4}^{k_3} \zeta[t, s] \left(\frac{\rho-t}{\rho-\bar{\theta}}\right)^{\eta-1} \left(\frac{k_4-s}{k_4-k_3}\right)^{\eta-1} \left(-\frac{dt}{\rho-\bar{\theta}}\right) \left(-\frac{ds}{k_4-k_3}\right) + \int_{\bar{\theta}}^{\rho} \int_{k_3}^{k_4} \zeta[v, w] \left(\frac{v-\bar{\theta}}{\rho-\bar{\theta}}\right)^{\eta-1} \left(\frac{w-k_3}{k_4-k_3}\right)^{\eta-1} \left(\frac{dv}{\rho-\bar{\theta}}\right) \left(\frac{dw}{k_4-k_3}\right)$$

$$h\left(\frac{1}{2}\right)\zeta\left[\frac{\bar{\theta}+\rho}{2}, \frac{k_3+k_4}{2}\right] \frac{1}{\eta^2} \leq \frac{1}{(\rho-\bar{\theta})^{\eta}(k_4-k_3)^{\eta}} \int_{\bar{\theta}}^{\rho} \int_{k_3}^{k_4} \zeta[t, s] (\rho-t)^{\eta-1} (k_4-s)^{\eta-1} dt ds + \frac{1}{(\rho-\bar{\theta})^{\eta}(k_4-k_3)^{\eta}} \int_{\bar{\theta}}^{\rho} \int_{k_3}^{k_4} \zeta[v, w] (v-\bar{\theta})^{\eta-1} (w-k_3)^{\eta-1} dv dw$$

$$\frac{h\left(\frac{1}{2}\right)}{2} \zeta\left[\frac{\bar{\theta}+\rho}{2}, \frac{k_3+k_4}{2}\right] \leq \frac{\gamma(\eta+1)}{2(\rho-\bar{\theta})^{\eta}(k_4-k_3)^{\eta}} \times \left[ \Xi_{(\bar{\theta}, k_3)}^{\eta} \zeta(\rho, k_4) + \Xi_{(\rho, k_4)}^{\eta} \zeta(\bar{\theta}, k_3) \right]. \quad (3.4)$$

For the second part of inequality, we take the two dimension h-Godunova-Levin convex function

$$\zeta[\theta x + (1-\theta)z, \theta y + (1-\theta)w] \leq \frac{\zeta(x, y)}{h(\theta)} + \frac{\zeta(z, w)}{h(1-\theta)} \quad (3.5)$$

and

$$\varsigma[(1-\theta)x+\theta z,(1-\theta)y+\theta w] \leq \frac{\varsigma(x,y)}{h(1-\theta)} + \frac{\varsigma(z,w)}{h(\theta)} \quad (3.6)$$

Adding the equations (3.5) and (3.6), then putting the values  $x = \theta\bar{\vartheta} + (1-\theta)\varrho, y = rk_3 + (1-r)k_4, z = (1-\theta)\bar{\vartheta} + \theta\varrho, w = (1-r)k_3 + rk_4$ , we have

$$\begin{aligned} & \varsigma[\theta\bar{\vartheta} + (1-\theta)\varrho, rk_3 + (1-r)k_4] + \varsigma[(1-\theta)\bar{\vartheta} + \theta\varrho, (1-r)k_3 + rk_4] \\ & \leq \varsigma[\bar{\vartheta} + \varrho, k_3 + k_4] \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right]. \end{aligned} \quad (3.7)$$

Multiplying both sides by  $\theta^{\eta-1}r^{\eta-1}$  and integrating with respect to  $\theta$ , and  $r$  on interval  $[0, 1]$ , we obtain

$$\begin{aligned} & \int_0^1 \int_0^1 \varsigma[\theta\bar{\vartheta} + (1-\theta)\varrho, rk_3 + (1-r)k_4] \theta^{\eta-1}r^{\eta-1} d\theta dr + \\ & \int_0^1 \int_0^1 \varsigma[(1-\theta)\bar{\vartheta} + \theta\varrho, (1-r)k_3 + rk_4] \theta^{\eta-1}r^{\eta-1} d\theta dr \\ & \leq \varsigma[\bar{\vartheta} + \varrho, k_3 + k_4] \int_0^1 \int_0^1 \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \theta^{\eta-1}r^{\eta-1} d\theta dr \\ & \frac{\gamma(\eta+1)}{(\varrho-\bar{\vartheta})^\eta(k_4-k_3)^\eta} \left[ \Xi_{(\bar{\vartheta},k_3)}^\eta \varsigma(\varrho, k_4) + \Xi_{(\varrho,k_4)}^\eta \varsigma(\bar{\vartheta}, k_3) \right] \leq \eta \varsigma[\bar{\vartheta} + \varrho, k_3 + k_4] \\ & \int_0^1 \int_0^1 \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \theta^{\eta-1}r^{\eta-1} d\theta dr. \end{aligned} \quad (3.8)$$

By combining equation (3.4) and (3.8), we reached the required inequality.  $\square$

**Corollary 3.1.1.** By suitable value of  $h(\theta) = \theta^s$  in theorem [3.1.1], we have Hermite-Hadamard fractional integral inequality for two dimensional  $s$ -Godunova-Levin function holds

$$\begin{aligned} & \frac{(\frac{1}{2})^s}{2} \varsigma\left[\frac{\bar{\vartheta} + \varrho}{2}, \frac{k_3 + k_4}{2}\right] \leq \frac{\gamma(\eta+1)}{2(\varrho-\bar{\vartheta})^\eta(k_4-k_3)^\eta} \left[ \Xi_{(\bar{\vartheta},k_3)}^\eta \varsigma(\varrho, k_4) + \Xi_{(\varrho,k_4)}^\eta \varsigma(\bar{\vartheta}, k_3) \right] \\ & \leq \eta \varsigma[\bar{\vartheta} + \varrho, k_3 + k_4] \int_0^1 \int_0^1 \left[ \frac{1}{(\theta)^s} + \frac{1}{(1-\theta)^s} \right] \theta^{\eta-1}r^{\eta-1} d\theta dr \end{aligned}$$



**Corollary 3.1.2.** By replacing the value of  $h(\theta) = 1$  in the theorem [3.1.1], we obtain the following Hermite-Hadamard type integral inequality for two dimension P function holds

$$\begin{aligned} \frac{1}{2} \zeta \left[ \frac{\bar{\theta} + \rho}{2}, \frac{k_3 + k_4}{2} \right] &\leq \frac{\gamma(\eta + 1)}{2(\rho - \bar{\theta})^\eta (k_4 - k_3)^\eta} \left[ \Xi_{(\bar{\theta}, k_3)+}^\eta \zeta(\rho, k_4) + \Xi_{(\rho, k_4)-}^\eta \zeta(\bar{\theta}, k_3) \right] \\ &\leq \frac{1}{\eta} \zeta \left[ \frac{\bar{\theta} + \rho, k_3 + k_4}{2} \right]. \end{aligned}$$

**Corollary 3.1.3.** By replacing the value  $h(\theta) = \frac{1}{\theta}$  in theorem [3.1.1], we have the following inequality

$$\begin{aligned} \zeta \left[ \frac{\bar{\theta} + \rho}{2}, \frac{k_3 + k_4}{2} \right] &\leq \frac{\gamma(\eta + 1)}{2(\rho - \bar{\theta})^\eta (k_4 - k_3)^\eta} \left[ \Xi_{(\bar{\theta}, k_3)+}^\eta \zeta(\rho, k_4) + \Xi_{(\rho, k_4)-}^\eta \zeta(\bar{\theta}, k_3) \right] \\ &\leq \frac{1}{\eta} \zeta \left[ \frac{\bar{\theta} + \rho, k_3 + k_4}{2} \right] \end{aligned}$$

**Corollary 3.1.4.** Applying theorem [3.1.1] for  $h(\theta) = \theta$ , we obtain the following inequality.

$$\begin{aligned} \frac{1}{4} \zeta \left[ \frac{\bar{\theta} + \rho}{2}, \frac{k_3 + k_4}{2} \right] &\leq \frac{\gamma(\eta + 1)}{2(\rho - \bar{\theta})^\eta (k_4 - k_3)^\eta} \left[ \Xi_{(\bar{\theta}, k_3)+}^\eta \zeta(\rho, k_4) + \Xi_{(\rho, k_4)-}^\eta \zeta(\bar{\theta}, k_3) \right] \\ &\leq \eta^2 \zeta \left[ \frac{\bar{\theta} + \rho, k_3 + k_4}{2} \right] \int_0^1 \int_0^1 \frac{\theta^{\eta-2}}{1-\theta} r^{\eta-1} d\theta dr \end{aligned}$$

**Corollary 3.1.5.** 1. If we putting the suitable value of  $h(\theta) = \frac{1}{\theta^s}$  in [3.1.1], we obtain the following inequality holds for s-convex function

$$\begin{aligned} 2^{s-1} \zeta \left[ \frac{\bar{\theta} + \rho}{2}, \frac{k_3 + k_4}{2} \right] &\leq \frac{\gamma(\eta + 1)}{2(\rho - \bar{\theta})^\eta (k_4 - k_3)^\eta} \left[ \Xi_{(\bar{\theta}, k_3)+}^\eta \zeta(\rho, k_4) + \Xi_{(\rho, k_4)-}^\eta \zeta(\bar{\theta}, k_3) \right] \\ &\leq \eta^2 \zeta \left[ \frac{\bar{\theta} + \rho, k_3 + k_4}{2} \right] \int_0^1 \int_0^1 \left[ \theta^s + (1-\theta)^s \right] \theta^{\eta-1} r^{\eta-1} d\theta dr. \end{aligned}$$

2. if we choosing  $\eta = 1$  in the theorem [3.1.1], we obtain

$$\begin{aligned} 2^{s-1} \zeta \left[ \frac{\bar{\theta} + \rho}{2}, \frac{k_3 + k_4}{2} \right] &\leq \frac{1}{(\rho - \bar{\theta})(k_4 - k_3)} \int_{\bar{\theta}}^{\rho} \int_{k_3}^{k_4} \zeta(\theta_1, \theta_2) d\theta_1 d\theta_2 \\ &\leq \zeta \left( \frac{\bar{\theta} + \rho, k_3 + k_4}{s+1} \right) \end{aligned}$$

## 3.2 Refined Trapezoid-Type Inequalities for Coordinated h-Godunova- Levin Pre-Invx Functions

In this section, we develop the lemma for coordinated h-Godunova-Levin pre-invex function, and discuss modified version of Trapezoid type integral inequalities by using the lemma. Moreover, we discuss some other inequalities in the form of corollaries.

**Lemma 3.2.1.** Let  $\varsigma : [\bar{\theta} + \xi_1(j\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3)] \rightarrow \mathbb{R}^2$  with  $\bar{\theta}, j\rho \in \mathbb{R}$ ,  $k_3, k_4 \in \mathbb{R}$ ,  $\varsigma \in L_1[\bar{\theta} + \xi_1(j\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3)]$  be a partial differentiable function where  $\Xi = [\bar{\theta} + \xi_1(j\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3)]$  be an open-invex set with respect to  $\xi_1 : \Xi \times \Xi \rightarrow \mathbb{R}$ ,  $\xi_2 : \Xi \times \Xi \rightarrow \mathbb{R}$  with  $\xi_1(j\rho, \bar{\theta}) > 0$ ,  $\xi_2(k_4, k_3) > 0$ , then the following result hold

$$\begin{aligned} & \frac{\varsigma(\bar{\theta}, k_3) + \varsigma(\bar{\theta} + \xi_1(j\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3))}{2} - \frac{\eta}{(j\rho - \bar{\theta})^\eta} \int_{\bar{\theta}}^{\bar{\theta} + \xi_1(j\rho, \bar{\theta})} (t - \bar{\theta})^{\eta-1} \varsigma(t, k_4) dt - \frac{\eta}{(k_4 - k_3)^\eta} \\ & \int_{k_3}^{k_3 + \xi_2(k_4, k_3)} (k_4 - t)^{\eta-1} \varsigma(\bar{\theta}, t) dt - \frac{\gamma(\eta+1)}{\xi_1(j\rho, \bar{\theta})^\eta \xi_2(k_4, k_3)^\eta} \left[ \Xi^{\eta+1, \eta} \varsigma(\bar{\theta} + \xi_1(j\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3)) + \right. \\ & \quad \left. \Xi^{\eta+1, \eta} (\bar{\theta} + \xi_1(j\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3)) - \varsigma(\bar{\theta}, k_3) \right] \\ & = \frac{\xi_1(j\rho, \bar{\theta}) \xi_2(k_4, k_3)}{2} \int_0^1 \int_0^1 [\theta^\eta r^\eta - (1-\theta)^\eta (1-r)^\eta] \frac{\partial^2 \varsigma}{\partial \theta \partial r} (\bar{\theta} + \theta \xi_1(j\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3)) d\theta dr \end{aligned}$$

*Proof.* Consider the integral

$$I_1 = \int_0^1 \int_0^1 [\theta^\eta r^\eta - (1-\theta)^\eta (1-r)^\eta] \frac{\partial^2 \varsigma}{\partial \theta \partial r} (\bar{\theta} + \theta \xi_1(j\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3)) d\theta dr$$

$$\begin{aligned} I_1 &= \int_0^1 \int_0^1 \theta^\eta r^\eta \frac{\partial^2 \varsigma}{\partial \theta \partial r} (\bar{\theta} + \theta \xi_1(j\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3)) d\theta dr - \\ & \int_0^1 \int_0^1 (1-\theta)^\eta (1-r)^\eta \frac{\partial^2 \varsigma}{\partial \theta \partial r} (\bar{\theta} + \theta \xi_1(j\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3)) d\theta dr \quad (3.9) \end{aligned}$$

Put the value of first and second integrals are  $I_2$  and  $I_3$  respectively, in the equation (3.9), and then consider  $I_2$ , we have

$$I_2 = \int_0^1 \int_0^1 \theta^\eta r^\eta \frac{\partial^2 \varsigma}{\partial \theta \partial r} (\bar{\theta} + \theta \xi_1(j\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3)) d\theta dr$$

$$I_2 = \int_0^1 \theta^\eta \left\{ \int_0^1 r^\eta \frac{\partial^2 \zeta}{\partial \theta \partial r} (\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3)) dr \right\} d\theta \quad (3.10)$$

Now, we putting the values of integral with respect to  $r$  is  $I_4$ , in equation (3.10), and then consider as follows

$$\begin{aligned} I_4 &= \int_0^1 r^\eta \frac{\partial^2 \zeta}{\partial \theta \partial r} (\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3)) dr \\ I_4 &= \left[ r^\eta \frac{\partial \zeta}{\partial \theta} \frac{(\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3))}{\xi_2(k_4, k_3)} \right]_0^1 - \eta \int_0^1 r^{\eta-1} \frac{\partial \zeta}{\partial \theta} \frac{(\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3))}{\xi_2(k_4, k_3)} dr \\ &= \left[ \frac{\partial \zeta}{\partial \theta} \frac{(\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3))}{\xi_2(k_4, k_3)} - \eta \int_0^1 r^{\eta-1} \frac{\partial \zeta}{\partial \theta} \frac{(\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3))}{\xi_2(k_4, k_3)} dr \right] \end{aligned} \quad (3.11)$$

Using the equation (3.11) in equation (3.10), then the integral  $I_2$  becomes

$$\begin{aligned} I_2 &= \frac{1}{\xi_2(k_4, k_3)} \left[ \int_0^1 \theta^\eta \frac{\partial \zeta}{\partial \theta} (\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3)) d\theta - \right. \\ &\quad \left. \eta \int_0^1 \int_0^1 \theta^\eta r^{\eta-1} \frac{\partial \zeta}{\partial \theta} (\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3)) d\theta dr \right] \end{aligned} \quad (3.12)$$

Putting the values of the integrals  $I_5$  and  $I_6$  in the right side of the equation (3.12) respectively, then consider the integral  $I_5$

$$\begin{aligned} I_5 &= \frac{1}{\xi_1(\rho, \bar{\theta})} \left[ \zeta(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3)) - \right. \\ &\quad \left. \eta \int_0^1 \theta^{\eta-1} \zeta(\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_4) d\theta \right] \end{aligned} \quad (3.13)$$

Putting the value of second integral  $I_7$  in the right side of the equation (3.13), and consider the simplification

$$\begin{aligned} I_7 &= \eta \int_0^1 \theta^{\eta-1} \zeta(\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_4) d\theta \\ I_7 &= \frac{\eta}{(\rho - \bar{\theta})^\eta} \int_{\bar{\theta}}^{\bar{\theta} + \xi_1(\rho, \bar{\theta})} (t - \bar{\theta})^{\eta-1} \zeta(t, k_4) dt \end{aligned} \quad (3.14)$$

Now, consider  $I_6$ , we have

$$I_6 = \eta \int_0^1 \int_0^1 \theta^\eta r^{\eta-1} \frac{\partial \zeta}{\partial \theta} (\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3)) d\theta dr \quad (3.15)$$

After simplification of the equation (3.15), we obtain

$$= \frac{\gamma(\eta+1)}{\xi_1(\rho, \bar{\theta})^\eta \xi_2(k_4, k_3)^\eta \xi_1(\rho, \bar{\theta})} \left[ \Xi^{\eta+1, \eta} \frac{(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3))}{(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3))} - \zeta(\bar{\theta}, k_3) \right] \quad (3.16)$$

Rearranging the values of the equations (3.13), (3.14) and (3.16), then the equation (3.10) becomes

$$\begin{aligned} I_2 &= \frac{1}{\xi_1(\rho, \bar{\theta}) \xi_2(k_4, k_3)} \left[ \left\{ \zeta(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3)) - \frac{\eta}{(\rho - \bar{\theta})^\eta} \right. \right. \\ &\quad \left. \left. \int_{\bar{\theta}}^{\bar{\theta} + \xi_1(\rho, \bar{\theta})} (t - \bar{\theta})^{\eta-1} \zeta(t, k_4) dt \right\} - \frac{\gamma(\eta+1)}{\xi_1(\rho, \bar{\theta})^\eta \xi_2(k_4, k_3)^\eta} \left\{ \Xi^{\eta+1, \eta} \frac{(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3))}{(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3))} - \zeta(\bar{\theta}, k_3) \right\} \right] \end{aligned}$$

Now, consider the integral  $I_3$  from equation (3.9), we have

$$\begin{aligned} I_3 &= \int_0^1 \int_0^1 (1-\theta)^\eta (1-r)^\eta \frac{\partial^2 \zeta}{\partial \theta \partial r} (\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3)) d\theta dr \\ I_3 &= \int_0^1 (1-\theta)^\eta \left\{ \int_0^1 (1-r)^\eta \frac{\partial^2 \zeta}{\partial \theta \partial r} (\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3)) dr \right\} d\theta \end{aligned} \quad (3.17)$$

Put the integral  $I_8$  in the right hand side of the equation, we have

$$\begin{aligned} I_8 &= \int_0^1 (1-r)^\eta \frac{\partial^2 \zeta}{\partial \theta \partial r} (\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3)) dr \\ &= - \left[ \frac{\partial \zeta}{\partial \theta} \frac{(\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3)}{\xi_2(k_4, k_3)} \right] - \eta \int_0^1 (1-r)^{\eta-1} \frac{\partial \zeta}{\partial \theta} \frac{(\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3))}{\xi_2(k_4, k_3)} dr \end{aligned}$$

$$= - \int_0^1 (1-\theta)^\eta \frac{\partial \zeta}{\partial \theta} \frac{(\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3) d\theta}{\xi_2(k_4, k_3)} - \eta \int_0^1 \int_0^1 (1-\theta)^\eta (1-r)^\eta \frac{\partial \zeta}{\partial \theta} \frac{(\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3))}{\xi_2(k_4, k_3)} d\theta dr \quad (3.18)$$

Putting the values of the integrals  $I_9$  and  $I_{10}$  in the right hand side of the equation (3.18), and then consider  $I_9$  and after simplification, we have

$$I_9 = \frac{1}{\xi_1(\rho, \bar{\theta}) \xi_2(k_4, k_3)} \left[ -\{\zeta(\bar{\theta}, k_3)\} + \eta \int_0^1 (1-\theta)^{\eta-1} \zeta(\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3) d\theta \right] \quad (3.19)$$

Putting the value of the  $I_{11}$  in the right side of the integral, we have

$$I_{11} = \eta \int_0^1 (1-\theta)^{\eta-1} \zeta(\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3) d\theta$$

$$I_{11} = \frac{\eta}{(\rho - \bar{\theta})^\eta} \int_{\bar{\theta}}^{\bar{\theta} + \xi(\rho, \bar{\theta})} (\rho - t)^{\eta-1} \zeta(t, k_3) dt \quad (3.20)$$

Now, consider  $I_{10}$  and after simplification, we have

$$= \frac{\gamma(\eta+1)}{\xi_1(\rho, \bar{\theta})^\eta \xi_2(k_4, k_3)^\eta \xi_1(\rho, \bar{\theta}) \xi_2(k_4, k_3)} \left[ \Xi_{(\bar{\theta}, k_3)+}^{\eta+1, \eta} \zeta(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3)) \right] \quad (3.21)$$

By combining (3.19), (3.20), (3.21), (3.18) in equation (3.17), we have

$$I_3 = \frac{1}{\xi_1(\rho, \bar{\theta}) \xi_2(k_4, k_3)} \left[ -\{\zeta(\bar{\theta}, k_3)\} + \frac{\eta}{(\rho - \bar{\theta})^\eta} \int_{\bar{\theta}}^{\bar{\theta} + \xi(\rho, \bar{\theta})} (\rho - t)^{\eta-1} \zeta(t, k_3) dt - \frac{\gamma(\eta+1)}{\xi_1(\rho, \bar{\theta})^\eta \xi_2(k_4, k_3)^\eta} \left( \Xi_{(\bar{\theta}, k_3)+}^{\eta+1, \eta} \zeta(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3)) \right) \right] \quad (3.22)$$

By combining (3.17) and (3.22) in equation in (3.9), we get the required result.  $\square$

**Theorem 3.2.1.** Let  $\zeta : \Xi = [\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3)] \rightarrow (0, \infty)$  with  $\Xi \in \mathbb{R}^2$ , be a partial differentiable function on  $\Xi$  and  $|\frac{\partial^2 \zeta}{\partial \theta \partial r}|$  be two dimensional (coordinated) h-

Godunova-Levin preinvex function on  $\Xi$ , then following integral inequality holds

$$\left| \frac{\zeta(\bar{\theta}, k_3) + \zeta(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3))}{2} - \frac{\eta}{(\rho - \bar{\theta})^\eta} \int_{\bar{\theta}}^{\bar{\theta} + \xi(\rho, \bar{\theta})} (\rho - t)^{\eta-1} \zeta(t, k_4) dt - \frac{\eta}{(\rho - \bar{\theta})^\eta} \int_{\bar{\theta}}^{\bar{\theta} + \xi(\rho, \bar{\theta})} (\rho - t)^{\eta-1} \zeta(t, k_3) dt - \frac{\gamma(\eta+1)}{\xi_1(\rho, \bar{\theta})^\eta \xi_2(k_4, k_3)^\eta} \left[ \Xi_{(\bar{\theta}, k_3)+}^{\eta+1, \eta} \zeta(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3)) + \Xi_{(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3))}^{\eta+1, \eta} \zeta(\bar{\theta}, k_3) \right] \right|$$

$$\leq \frac{\xi_1(\rho, \bar{\theta}) \xi_2(k_4, k_3)}{2} \left[ \left| \frac{\partial^2 \zeta(\bar{\theta}, k_3)}{\partial \theta \partial r} \right| + \left| \frac{\partial^2 \zeta(\bar{\theta}, k_4)}{\partial \theta \partial r} \right| + \left| \frac{\partial^2 \zeta(\rho, k_3)}{\partial \theta \partial r} \right| + \left| \frac{\partial^2 \zeta(\rho, k_4)}{\partial \theta \partial r} \right| \right]$$

$$\times \int_0^1 \int_0^1 \left| \theta^\eta r^\eta - (1-\theta)^\eta (1-r)^\eta \right| \frac{1}{h(\bar{\theta} r)} d\theta dr$$

*Proof.* Consider the lemma [3.2.1]

$$\left| \frac{\zeta(\bar{\theta}, k_3) + \zeta(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3))}{2} - \frac{\eta}{(\rho - \bar{\theta})^\eta} \int_{\bar{\theta}}^{\bar{\theta} + \xi(\rho, \bar{\theta})} (\rho - t)^{\eta-1} \zeta(t, k_4) dt - \frac{\eta}{(\rho - \bar{\theta})^\eta} \int_{\bar{\theta}}^{\bar{\theta} + \xi(\rho, \bar{\theta})} (\rho - t)^{\eta-1} \zeta(t, k_3) dt - \frac{\gamma(\eta+1)}{\xi_1(\rho, \bar{\theta})^\eta \xi_2(k_4, k_3)^\eta} \left[ \Xi_{(\bar{\theta}, k_3)+}^{\eta+1, \eta} \zeta(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3)) + \Xi_{(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3))}^{\eta+1, \eta} \zeta(\bar{\theta}, k_3) \right] \right|$$

$$= \left| \frac{\xi_1(\rho, \bar{\theta}) \xi_2(k_4, k_3)}{2} \int_0^1 \int_0^1 [\theta^\eta r^\eta - (1-\theta)^\eta (1-r)^\eta] \frac{\partial^2 \zeta}{\partial \theta \partial r} (\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3)) d\theta dr \right|$$

$$\leq \frac{\xi_1(\rho, \bar{\theta}) \xi_2(k_4, k_3)}{2} \int_0^1 \int_0^1 \left| \theta^\eta r^\eta - (1-\theta)^\eta (1-r)^\eta \right| \left| \frac{\partial^2 \zeta}{\partial \theta \partial r} (\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3)) \right| d\theta dr \quad (3.23)$$

Applying the definition of h-Godunova-Levin coordinated pre-invex function in equation (3.23), we have

$$\leq \frac{\xi_1(\rho, \bar{\theta}) \xi_2(k_4, k_3)}{2} \int_0^1 \int_0^1 \left| \theta^\eta r^\eta - (1-\theta)^\eta (1-r)^\eta \right| \left| \frac{\frac{\partial^2 \zeta(\bar{\theta}, k_3)}{\partial \theta \partial r}}{h[(1-\theta)(1-r)]} + \frac{\frac{\partial^2 \zeta(\bar{\theta}, k_4)}{\partial \theta \partial r}}{h[(1-\theta)(r)]} + \frac{\frac{\partial^2 \zeta(\rho, k_3)}{\partial \theta \partial r}}{h[(\theta)(1-r)]} + \frac{\frac{\partial^2 \zeta(\rho, k_4)}{\partial \theta \partial r}}{h[(\theta)(r)]} \right| d\theta dr$$



$$\begin{aligned} &\leq \frac{\xi_1(\rho, \bar{\theta})\xi_2(k_4, k_3)}{2} \int_0^1 \int_0^1 \left| \theta^\eta r^\eta - (1-\theta)^\eta (1-r)^\eta \right| \left[ \frac{\left| \frac{\partial^2 \zeta(\bar{\theta}, k_3)}{\partial \theta \partial r} \right|}{h[(1-\theta)(1-r)]} + \right. \\ &\quad \left. \frac{\left| \frac{\partial^2 \zeta(\bar{\theta}, k_4)}{\partial \theta \partial r} \right|}{h[(\theta)(1-r)]} + \frac{\left| \frac{\partial^2 \zeta(\rho, k_3)}{\partial \theta \partial r} \right|}{h[(\theta)(r)]} \right] d\theta dr \\ &= \frac{\xi_1(\rho, \bar{\theta})\xi_2(k_4, k_3)}{2} \left[ \left| \frac{\partial^2 \zeta(\bar{\theta}, k_3)}{\partial \theta \partial r} \right| + \left| \frac{\partial^2 \zeta(\bar{\theta}, k_4)}{\partial \theta \partial r} \right| + \left| \frac{\partial^2 \zeta(\rho, k_3)}{\partial \theta \partial r} \right| \right. \\ &\quad \left. + \left| \frac{\partial^2 \zeta(\rho, k_4)}{\partial \theta \partial r} \right| \right] \int_0^1 \int_0^1 \left| \theta^\eta r^\eta - (1-\theta)^\eta (1-r)^\eta \right| \frac{1}{h(\theta r)} d\theta dr. \end{aligned}$$

Which are the required result.  $\square$

**Corollary 3.2.1.** If we replace  $\eta = 1$  in theorem [3.2.1], we have the following inequality

$$\begin{aligned} &\left| \frac{\zeta(\bar{\theta}, k_3) + \zeta(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3))}{2} - \frac{1}{\xi_1(\rho, \bar{\theta})} \int_{\bar{\theta}}^{\bar{\theta} + \xi_1(\rho, \bar{\theta})} \zeta(t, k_4) dt - \right. \\ &\quad \left. \frac{k_4 - k_3}{\xi_1(\rho, \bar{\theta})^2 \xi_2(k_4, k_3)^2} \int_{\bar{\theta}}^{\bar{\theta} + \xi_1(\rho, \bar{\theta})} (t - \bar{\theta}) \zeta(t) dt - \frac{1}{\xi_1(\rho, \bar{\theta})} \int_{\bar{\theta}}^{\bar{\theta} + \xi_1(\rho, \bar{\theta})} \zeta(t, k_3) dt - \right. \\ &\quad \left. \frac{k_4 - k_3}{\xi_1(\rho, \bar{\theta}) \xi_2(k_4, k_3)} \int_{\bar{\theta}}^{\bar{\theta} + \xi_1(\rho, \bar{\theta})} (\rho - t) \zeta(t) dt \right| \leq \frac{\xi_1(\rho, \bar{\theta})\xi_2(k_4, k_3)}{2} \left[ \left| \frac{\partial^2 \zeta(\bar{\theta}, k_3)}{\partial \theta \partial r} \right| + \right. \\ &\quad \left. \left| \frac{\partial^2 \zeta(\bar{\theta}, k_4)}{\partial \theta \partial r} \right| + \left| \frac{\partial^2 \zeta(\rho, k_3)}{\partial \theta \partial r} \right| + \left| \frac{\partial^2 \zeta(\rho, k_4)}{\partial \theta \partial r} \right| \right] \int_0^1 \int_0^1 |\theta r - (1-\theta)(1-r)| \frac{1}{h(\theta r)} d\theta dr \end{aligned}$$

**Corollary 3.2.2.** If we replace  $\xi_1(\rho, \bar{\theta}) = \rho - \bar{\theta}$  and  $\xi_2(k_4, k_3) = k_4 - k_3$  in theorem [3.2.1] then we obtain the following inequality

$$\begin{aligned} &\left| \frac{\zeta(\bar{\theta}, k_3) + \zeta(\rho, k_4)}{2} - \frac{\eta}{(\rho - \bar{\theta})^\eta} \int_{\bar{\theta}}^{\rho} (t - \bar{\theta})^{\eta-1} \zeta(t, k_4) dt - \frac{\eta}{(\rho - \bar{\theta})^\eta} \right. \\ &\quad \left. \int_{\bar{\theta}}^{\rho} (\rho - t)^{\eta-1} \zeta(t, k_3) dt - \frac{\gamma(\eta+1)}{(\rho - \bar{\theta})^\eta (k_4 - k_3)^\eta} \left[ \Xi^{\eta+1, \eta}_{(\bar{\theta}, k_3)+} \zeta(\rho, k_4) + \Xi^{\eta+1, \eta}_{(\rho, k_4)-} \zeta(\bar{\theta}, k_3) \right] \right| \\ &\leq \frac{(\rho - \bar{\theta})(k_4 - k_3)}{2} \left[ \left| \frac{\partial^2 \zeta(\bar{\theta}, k_3)}{\partial \theta \partial r} \right| + \left| \frac{\partial^2 \zeta(\bar{\theta}, k_4)}{\partial \theta \partial r} \right| + \left| \frac{\partial^2 \zeta(\rho, k_3)}{\partial \theta \partial r} \right| + \left| \frac{\partial^2 \zeta(\rho, k_4)}{\partial \theta \partial r} \right| \right] \times \\ &\quad \int_0^1 \int_0^1 \left| \theta^\eta r^\eta - (1-\theta)^\eta (1-r)^\eta \right| \frac{1}{h(\theta r)} d\theta dr \end{aligned}$$

**Theorem 3.2.2.** Let  $\zeta : \Xi = [\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3)] \rightarrow (0, \infty)$  with  $\Xi \in \mathbb{R}^2$ , be a partial differentiable real valued function on  $\Xi$  and  $|\frac{\partial^2 \zeta}{\partial \theta \partial r}|^q$  be two dimensional (coor-

minated)  $h$ -Godunova-Levin pre-invex function on  $\Xi$  for  $p > 1$  and  $q = \frac{p}{p-1}$ , then the following inequality holds

$$\begin{aligned} &\left| \frac{\zeta(\bar{\theta}, k_3) + \zeta(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3))}{2} - \frac{\eta}{(\rho - \bar{\theta})^\eta} \int_{\bar{\theta}}^{\bar{\theta} + \xi_1(\rho, \bar{\theta})} (t - \bar{\theta})^{\eta-1} \zeta(t, k_4) dt - \right. \\ &\quad \left. \frac{\eta}{(\rho - \bar{\theta})^\eta} \int_{\bar{\theta}}^{\bar{\theta} + \xi_1(\rho, \bar{\theta})} (\rho - t)^{\eta-1} \zeta(t, k_3) dt - \frac{\gamma(\eta+1)}{\xi_1(\rho, \bar{\theta})^\eta \xi_2(k_4, k_3)^\eta} \right. \\ &\quad \left. \left[ \Xi^{\eta+1, \eta}_{(\bar{\theta}, k_3)+} \zeta(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3)) + \Xi^{\eta+1, \eta}_{(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3)) -} \zeta(\bar{\theta}, k_3) \right] \right| \\ &\leq \frac{\xi_1(\rho, \bar{\theta})\xi_2(k_4, k_3)}{2} \left[ \left| \frac{\partial^2 \zeta(\bar{\theta}, k_3)}{\partial \theta \partial r} \right|^q + \left| \frac{\partial^2 \zeta(\bar{\theta}, k_4)}{\partial \theta \partial r} \right|^q + \left| \frac{\partial^2 \zeta(\rho, k_3)}{\partial \theta \partial r} \right|^q + \right. \\ &\quad \left. \left| \frac{\partial^2 \zeta(\rho, k_4)}{\partial \theta \partial r} \right|^q \right]^{\frac{1}{q}} \left( \int_0^1 \int_0^1 |\theta^\eta r^\eta - (1-\theta)^\eta (1-r)^\eta|^p d\theta dr \right)^{\frac{1}{p}} \left( \int_0^1 \int_0^1 \frac{1}{h(\theta r)} d\theta dr \right)^{\frac{1}{q}} \end{aligned}$$

*Proof.* Consider the lemma [3.2.1], we have

$$\begin{aligned} &\left| \frac{\zeta(\bar{\theta}, k_3) + \zeta(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3))}{2} - \frac{\eta}{(\rho - \bar{\theta})^\eta} \int_{\bar{\theta}}^{\bar{\theta} + \xi_1(\rho, \bar{\theta})} (t - \bar{\theta})^{\eta-1} \zeta(t) dt - \right. \\ &\quad \left. \frac{\eta}{(\rho - \bar{\theta})^\eta} \int_{\bar{\theta}}^{\bar{\theta} + \xi_1(\rho, \bar{\theta})} (\rho - t)^{\eta-1} \zeta(t) dt - \frac{\gamma(\eta+1)}{\xi_1(\rho, \bar{\theta})^\eta \xi_2(k_4, k_3)^\eta} \right. \\ &\quad \left. \left[ \Xi^{\eta+1, \eta}_{(\bar{\theta}, k_3)+} \zeta(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3)) + \Xi^{\eta+1, \eta}_{(\bar{\theta} + \xi_1(\rho, \bar{\theta}), k_3 + \xi_2(k_4, k_3)) -} \zeta(\bar{\theta}, k_3) \right] \right| \\ &= \left| \frac{\xi_1(\rho, \bar{\theta})\xi_2(k_4, k_3)}{2} \int_0^1 \int_0^1 [\theta^\eta r^\eta - (1-\theta)^\eta (1-r)^\eta] \frac{\partial^2 \zeta}{\partial \theta \partial r} \right. \\ &\quad \left. (\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3)) d\theta dr \right|. \end{aligned} \quad (3.24)$$

After simplification the equation (3.24), we obtain

$$\leq \frac{\xi_1(\rho, \bar{\theta})\xi_2(k_4, k_3)}{2} \int_0^1 \int_0^1 \left| \theta^\eta r^\eta - (1-\theta)^\eta (1-r)^\eta \right| \left| \frac{\partial^2 \zeta}{\partial \theta \partial r} (\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3)) \right| d\theta dr.$$

Applying the Holder's inequality, we obtain

$$\begin{aligned} &\leq \frac{\xi_1(\rho, \bar{\theta})\xi_2(k_4, k_3)}{2} \left( \int_0^1 \int_0^1 |\theta^\eta r^\eta - (1-\theta)^\eta (1-r)^\eta|^p d\theta dr \right)^{\frac{1}{p}} \times \\ &\quad \left( \int_0^1 \int_0^1 \left| \frac{\partial^2 \zeta}{\partial \theta \partial r} (\bar{\theta} + \theta \xi_1(\rho, \bar{\theta}), k_3 + r \xi_2(k_4, k_3)) \right|^q d\theta dr \right)^{\frac{1}{q}} \end{aligned} \quad (3.25)$$



consider the second integral of right hand side of the equation (3.25), and applying the definition of h-Godunova-Levin coordinated pre-invex function, we have

$$\begin{aligned} & \int_0^1 \int_0^1 \left| \frac{\partial^2 \zeta}{\partial \theta \partial r} (\bar{\theta} + \theta \xi_1(\varphi, \bar{\theta}), k_3 + r \xi_2(k_4, k_3)) \right|^q d\theta dr \\ & \leq \int_0^1 \int_0^1 \left[ \frac{\left| \frac{\partial^2 \zeta(\bar{\theta}, k_3)}{\partial \theta \partial r} \right|^q}{h[(1-\theta)(1-r)]} + \frac{\left| \frac{\partial^2 \zeta(\bar{\theta}, k_4)}{\partial \theta \partial r} \right|^q}{h[(1-\theta)(r)]} + \frac{\left| \frac{\partial^2 \zeta(\varphi, k_3)}{\partial \theta \partial r} \right|^q}{h[(\theta)(1-r)]} + \frac{\left| \frac{\partial^2 \zeta(\varphi, k_4)}{\partial \theta \partial r} \right|^q}{h[(\theta)(r)]} \right] d\theta dr \\ & \leq \left[ \left| \frac{\partial^2 \zeta}{\partial \theta \partial r}(\bar{\theta}, k_3) \right|^q + \left| \frac{\partial^2 \zeta}{\partial \theta \partial r}(\bar{\theta}, k_4) \right|^q + \left| \frac{\partial^2 \zeta}{\partial \theta \partial r}(\varphi, k_3) \right|^q + \left| \frac{\partial^2 \zeta}{\partial \theta \partial r}(\varphi, k_4) \right|^q \right] \\ & \quad \int_0^1 \int_0^1 \frac{1}{h[(\theta)(r)]} d\theta dr \end{aligned} \quad (3.26)$$

using equation (3.26) in equation (3.25), we get the required result.  $\square$

## Modification of Inequalities through Riemann–Liouville Fuzzy Fractional Integral Operators

This study introduces a new class of convex and non-convex (pre-invex) fuzzy interval-valued functions (FIVFs) that incorporate an arbitrary auxiliary function hhh. These newly defined functions are termed h-convex FIVFs and h-pre-invex FIVFs, respectively. By employing this generalized framework in conjunction with fuzzy order relations, a series of fractional integral inequalities of Hermite–Hadamard and trapezoidal types are established. The results not only extend existing inequalities to the fuzzy fractional domain but also provide several notable corollaries that reinforce and generalize the main theoretical findings.

### 4.1 Analysis of Hermite–Hadamard (HH) Type Inequalities for h-Godunova–Levin Convex Fuzzy Interval-Valued Functions

In this section, we develop and analyze a new class of convex and non-convex (pre-invex) fuzzy interval-valued functions using the auxiliary hhh-function within the framework of Riemann–Liouville fractional operators. Based on this extended class—referred to as h-Godunova–Levin convex FIVFs—we derive several Hermite–Hadamard type inequalities that capture the fractional and fuzzy characteristics of the defined functions. Furthermore, we discuss specific applications of these results in the form of corollaries, highlighting their analytical significance and potential use in broader mathematical modeling contexts.

**Theorem 4.1.1.** Let  $h: (0, 1) \rightarrow \mathbb{R}$  be a positive real valued function and  $\mathfrak{S} \in L_1[\bar{\vartheta}, \varrho]$ , such that  $h(\theta) \neq 0$ ,  $\mathfrak{S}: [\bar{\vartheta}, \varrho] \rightarrow \mathbb{R}$  be  $h$ -Godunova-Levin convex FIVE, where  $0 < \bar{\vartheta} < \varrho$ , then the following fuzzy fractional integral inequality holds

$$\begin{aligned} \frac{h(\frac{1}{2})}{2} \mathfrak{S}\left(\frac{\bar{\vartheta} + \varrho}{2}\right) &\leq \frac{\Gamma(\mathfrak{J}+1)}{2(\varrho - \bar{\vartheta})^{\mathfrak{J}}} \left[ \Lambda_{\bar{\vartheta}}^{\mathfrak{J}} \mathfrak{S}(\varrho) + \Lambda_{\varrho}^{\mathfrak{J}} \mathfrak{S}(\bar{\vartheta}) \right] \\ &\leq \frac{\mathfrak{S}(\bar{\vartheta}) + \mathfrak{S}(\varrho)}{2} \int_0^1 \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \theta^{\mathfrak{J}-1} d\theta. \end{aligned}$$

*Proof.* Consider the definition of  $h$ -Godunova-Levin convex function over the closed interval  $[\bar{\vartheta}, \varrho]$ , and  $x, y \in [\bar{\vartheta}, \varrho]$ , we have

$$\mathfrak{S}(\delta x + (1-\delta)y) \leq \frac{\mathfrak{S}(x)}{h(\delta)} + \frac{\mathfrak{S}(y)}{h(1-\delta)} \quad (4.1)$$

Putting the values  $x = \theta\bar{\vartheta} + (1-\theta)\varrho, y = (1-\theta)\bar{\vartheta} + \theta\varrho$  and  $\delta = \frac{1}{2}$ , we have

$$h\left(\frac{1}{2}\right) \mathfrak{S}\left(\frac{\bar{\vartheta} + \varrho}{2}\right) \leq [\mathfrak{S}(\theta\bar{\vartheta} + (1-\theta)\varrho) + \mathfrak{S}((1-\theta)\bar{\vartheta} + \theta\varrho)]. \quad (4.2)$$

If we take for every  $\theta \in [0, 1]$ , we have

$$h\left(\frac{1}{2}\right) \mathfrak{S}_*\left(\frac{\bar{\vartheta} + \varrho}{2}\right) \leq [\mathfrak{S}_*(\theta\bar{\vartheta} + (1-\theta)\varrho) + \mathfrak{S}_*((1-\theta)\bar{\vartheta} + \theta\varrho)]. \quad (4.3)$$

Multiplying  $\theta^{\mathfrak{J}-1}$  in the equation (5.3) and then integrating with respect to  $\theta$  over the interval  $[0, 1]$ , we obtain

$$h\left(\frac{1}{2}\right) \mathfrak{S}_*\left(\frac{\bar{\vartheta} + \varrho}{2}\right) \int_0^1 \theta^{\mathfrak{J}-1} d\theta \leq \int_0^1 \theta^{\mathfrak{J}-1} \mathfrak{S}_*(\theta\bar{\vartheta} + (1-\theta)\varrho) d\theta + \int_0^1 \theta^{\mathfrak{J}-1} \mathfrak{S}_*((1-\theta)\bar{\vartheta} + \theta\varrho) d\theta.$$

Now by substituting:  $w = \theta\bar{\vartheta} + (1-\theta)\varrho$  and  $t = (1-\theta)\bar{\vartheta} + \theta\varrho$ ,

$$h\left(\frac{1}{2}\right) \mathfrak{S}_*\left(\frac{\bar{\vartheta} + \varrho}{2}\right) \frac{1}{2} \leq \int_{\varrho}^{\bar{\vartheta}} \left(\frac{\varrho - w}{\varrho - \bar{\vartheta}}\right)^{\mathfrak{J}-1} \mathfrak{S}_*(w) \frac{dw}{\varrho - \bar{\vartheta}} + \int_{\bar{\vartheta}}^{\varrho} \left(\frac{t - \bar{\vartheta}}{\varrho - \bar{\vartheta}}\right)^{\mathfrak{J}-1} \mathfrak{S}_*(w) \frac{dt}{\varrho - \bar{\vartheta}}.$$

$$\frac{h(\frac{1}{2})}{2} \mathfrak{S}_*\left(\frac{\bar{\vartheta} + \varrho}{2}\right) \leq \frac{\Gamma(\mathfrak{J}+1)}{2(\varrho - \bar{\vartheta})^{\mathfrak{J}}} [\Lambda_{\bar{\vartheta}}^{\mathfrak{J}} \mathfrak{S}_*(\varrho) + \Lambda_{\varrho}^{\mathfrak{J}} \mathfrak{S}_*(\bar{\vartheta})]. \quad (4.4)$$

Similarly, we solve for upper FIVE for  $\mathfrak{S}^*$ , we obtain

$$\frac{h(\frac{1}{2})}{2} \mathfrak{S}^*\left(\frac{\bar{\vartheta} + \varrho}{2}\right) \leq \frac{\Gamma(\mathfrak{J}+1)}{2(\varrho - \bar{\vartheta})^{\mathfrak{J}}} [\Lambda_{\bar{\vartheta}}^{\mathfrak{J}} \mathfrak{S}^*(\varrho) + \Lambda_{\varrho}^{\mathfrak{J}} \mathfrak{S}^*(\bar{\vartheta})]. \quad (4.5)$$

From equation (5.5) and (4.5)

$$\begin{aligned} \frac{h(\frac{1}{2})}{2} \left[ \mathfrak{S}_*\left(\frac{\bar{\vartheta} + \varrho}{2}\right), \mathfrak{S}^*\left(\frac{\bar{\vartheta} + \varrho}{2}\right) \right] &\leq \\ \frac{\Gamma(\mathfrak{J}+1)}{2(\varrho - \bar{\vartheta})^{\mathfrak{J}}} \left[ \{ \Lambda_{\bar{\vartheta}}^{\mathfrak{J}} \mathfrak{S}_*(\varrho) + \Lambda_{\varrho}^{\mathfrak{J}} \mathfrak{S}_*(\bar{\vartheta}) \}, \{ \Lambda_{\bar{\vartheta}}^{\mathfrak{J}} \mathfrak{S}^*(\varrho) + \Lambda_{\varrho}^{\mathfrak{J}} \mathfrak{S}^*(\bar{\vartheta}) \} \right] \end{aligned}$$

That is:

$$\frac{h(\frac{1}{2})}{2} \mathfrak{S}\left(\frac{\bar{\vartheta} + \varrho}{2}\right) \leq \frac{\Gamma(\mathfrak{J}+1)}{2(\varrho - \bar{\vartheta})^{\mathfrak{J}}} [\Lambda_{\bar{\vartheta}}^{\mathfrak{J}} \mathfrak{S}(\varrho) + \Lambda_{\varrho}^{\mathfrak{J}} \mathfrak{S}(\bar{\vartheta})] \quad (4.6)$$

Now, we consider  $h$ -Godunova-Levin convexity of  $\mathfrak{S}$ , for second part of inequality,

$$\mathfrak{S}(\theta\bar{\vartheta} + (1-\theta)\varrho) \leq \frac{\mathfrak{S}(\bar{\vartheta})}{h(\theta)} + \frac{\mathfrak{S}(\varrho)}{h(1-\theta)}, \quad (4.7)$$

and

$$\mathfrak{S}((1-\theta)\bar{\vartheta} + \theta\varrho) \leq \frac{\mathfrak{S}(\bar{\vartheta})}{h(1-\theta)} + \frac{\mathfrak{S}(\varrho)}{h(\theta)}. \quad (4.8)$$

By adding equation (5.6) and (5.7), we get

$$\mathfrak{S}(\theta\bar{\vartheta} + (1-\theta)\varrho) + \mathfrak{S}((1-\theta)\bar{\vartheta} + \theta\varrho) \leq (\mathfrak{S}(\bar{\vartheta}) + \mathfrak{S}(\varrho)) \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \quad (4.9)$$

If we take for every  $\theta \in [0, 1]$ , we have

$$\begin{aligned} \mathfrak{S}_*(\theta\bar{\vartheta} + (1-\theta)\varrho) + \mathfrak{S}_*((1-\theta)\bar{\vartheta} + \theta\varrho) &\leq (\mathfrak{S}_*(\bar{\vartheta}) + \mathfrak{S}_*(\varrho)) \\ \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right]. \end{aligned} \quad (4.10)$$

By multiplying  $\theta^{\mathfrak{J}-1}$  on both sides of equation (5.9) and integrating inequality with respect to  $\theta$  over interval  $[0, 1]$ , we get

$$\begin{aligned} \int_0^1 \theta^{\mathfrak{J}-1} \mathfrak{S}_*(\theta\bar{\vartheta} + (1-\theta)\varrho) d\theta + \int_0^1 \theta^{\mathfrak{J}-1} \mathfrak{S}_*((1-\theta)\bar{\vartheta} + \theta\varrho) d\theta \\ \leq (\mathfrak{S}_*(\bar{\vartheta}) + \mathfrak{S}_*(\varrho)) \int_0^1 \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \theta^{\mathfrak{J}-1} d\theta. \end{aligned}$$

After simplification, we obtain

$$\begin{aligned} & \frac{\Gamma(\mathfrak{J}+1)}{(\rho-\bar{\theta})^{\mathfrak{J}}} [\Lambda_{\bar{\theta}}^{\mathfrak{J}} \mathfrak{S}_*(\rho) + \Lambda_{\rho}^{\mathfrak{J}} \mathfrak{S}_*(\bar{\theta})] \\ & \leq \mathfrak{J} [\mathfrak{S}_*(\bar{\theta}) + \mathfrak{S}_*(\rho)] \int_0^1 \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \theta^{\mathfrak{J}-1} d\theta. \end{aligned} \quad (4.11)$$

Similarly, we solve the upper FIVF for  $\mathfrak{S}^*$ , we obtain

$$\begin{aligned} & \frac{\Gamma(\mathfrak{J}+1)}{(\rho-\bar{\theta})^{\mathfrak{J}}} [\Lambda_{\bar{\theta}}^{\mathfrak{J}} \mathfrak{S}^*(\rho) + \Lambda_{\rho}^{\mathfrak{J}} \mathfrak{S}^*(\bar{\theta})] \leq \mathfrak{J} [\mathfrak{S}^*(\bar{\theta}) + \\ & \mathfrak{S}^*(\rho)] \int_0^1 \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \theta^{\mathfrak{J}-1} d\theta. \end{aligned} \quad (4.12)$$

Combining equation (5.10) and (5.12), we get the following inequality.

$$\begin{aligned} & \frac{\Gamma(\mathfrak{J}+1)}{(\rho-\bar{\theta})^{\mathfrak{J}}} \left[ \{\Lambda_{\bar{\theta}}^{\mathfrak{J}} \mathfrak{S}_*(\rho) + \Lambda_{\rho}^{\mathfrak{J}} \mathfrak{S}_*(\bar{\theta})\}, \{\Lambda_{\bar{\theta}}^{\mathfrak{J}} \mathfrak{S}^*(\rho) + \Lambda_{\rho}^{\mathfrak{J}} \mathfrak{S}^*(\bar{\theta})\} \right] \\ & \leq \mathfrak{J} [\mathfrak{S}_*(\bar{\theta}) + \mathfrak{S}_*(\rho)], [\mathfrak{S}^*(\bar{\theta}) + \\ & \mathfrak{S}^*(\rho)] \int_0^1 \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \theta^{\mathfrak{J}-1} d\theta. \end{aligned} \quad (4.13)$$

That is

$$\begin{aligned} & \frac{\Gamma(\mathfrak{J}+1)}{2(\rho-\bar{\theta})^{\mathfrak{J}}} [\Lambda_{\bar{\theta}}^{\mathfrak{J}} \mathfrak{S}(\rho) + \Lambda_{\rho}^{\mathfrak{J}} \mathfrak{S}(\bar{\theta})] \leq \frac{\mathfrak{S}(\bar{\theta}) \tilde{+} \mathfrak{S}(\rho)}{2} \int_0^1 \times \\ & \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \theta^{\mathfrak{J}-1} d\theta \end{aligned} \quad (4.14)$$

From equation (4.6) and (5.13)

$$\begin{aligned} & \frac{h(\frac{1}{2})}{2} \mathfrak{S} \left( \frac{\bar{\theta} + \rho}{2} \right) \leq \frac{\Gamma(\mathfrak{J}+1)}{2(\rho-\bar{\theta})^{\mathfrak{J}}} [\Lambda_{\bar{\theta}}^{\mathfrak{J}} \mathfrak{S}(\rho) \tilde{+} \Lambda_{\rho}^{\mathfrak{J}} \mathfrak{S}(\bar{\theta})] \leq \frac{\mathfrak{S}(\bar{\theta}) \tilde{+} \mathfrak{S}(\rho)}{2} \int_0^1 \times \\ & \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \theta^{\mathfrak{J}-1} d\theta \end{aligned}$$

which is our require result.  $\square$

**Corollary 4.1.1.** By putting the value  $h(\theta) = \theta^s$  in theorem 5.1.1, then we obtain the

(H – H)-type inequality for s-Godunova-Levin convex fuzzy IVF, as given below

$$\begin{aligned} & \frac{(\frac{1}{2})^s}{2} \mathfrak{S} \left( \frac{\bar{\theta} + \rho}{2} \right) \leq \frac{\Gamma(\mathfrak{J}+1)}{2(\rho-\bar{\theta})^{\mathfrak{J}}} [\Lambda_{\bar{\theta}}^{\mathfrak{J}} \mathfrak{S}(\rho) \tilde{+} \Lambda_{\rho}^{\mathfrak{J}} \mathfrak{S}(\bar{\theta})] \\ & \leq \frac{\mathfrak{S}(\bar{\theta}) \tilde{+} \mathfrak{S}(\rho)}{2} \int_0^1 \left[ \frac{1}{(\theta)^s} + \frac{1}{(1-\theta)^s} \right] \theta^{\mathfrak{J}-1} d\theta. \end{aligned}$$

**Corollary 4.1.2.** By replacing  $h(\theta) = 1$  in theorem 5.1.1, we have the following inequality

$$\frac{1}{2} \mathfrak{S} \left( \frac{\bar{\theta} + \rho}{2} \right) \leq \frac{\Gamma(\mathfrak{J}+1)}{2(\rho-\bar{\theta})^{\mathfrak{J}}} [\Lambda_{\bar{\theta}}^{\mathfrak{J}} \mathfrak{S}(\rho) \tilde{+} \Lambda_{\rho}^{\mathfrak{J}} \mathfrak{S}(\bar{\theta})] \leq [\mathfrak{S}(\bar{\theta}) \tilde{+} \mathfrak{S}(\rho)].$$

**Corollary 4.1.3.** By replacing the value  $h(\theta) = \frac{1}{\theta}$  in theorem 5.1.1, we obtain the following inequality

$$\mathfrak{S} \left( \frac{\bar{\theta} + \rho}{2} \right) \leq \frac{\Gamma(\mathfrak{J}+1)}{2(\rho-\bar{\theta})^{\mathfrak{J}}} [\Lambda_{\bar{\theta}}^{\mathfrak{J}} \mathfrak{S}(\rho) \tilde{+} \Lambda_{\rho}^{\mathfrak{J}} \mathfrak{S}(\bar{\theta})] \leq \frac{\mathfrak{S}(\bar{\theta}) \tilde{+} \mathfrak{S}(\rho)}{2}.$$

**Corollary 4.1.4.** If we take the value  $h(\theta) = \theta$  in theorem 5.1.1, then obtain the integral inequality as follows

$$\frac{1}{4} \mathfrak{S} \left( \frac{\bar{\theta} + \rho}{2} \right) \leq \frac{\Gamma(\mathfrak{J}+1)}{2(\rho-\bar{\theta})^{\mathfrak{J}}} [\Lambda_{\bar{\theta}}^{\mathfrak{J}} \mathfrak{S}(\rho) \tilde{+} \Lambda_{\rho}^{\mathfrak{J}} \mathfrak{S}(\bar{\theta})] \leq \frac{\mathfrak{S}(\bar{\theta}) \tilde{+} \mathfrak{S}(\rho)}{2} \int_0^1 \frac{\theta^{\mathfrak{J}-2}}{1-\theta} d\theta.$$

**Corollary 4.1.5.** 1. by choosing the value  $h(\theta) = \frac{1}{\theta^s}$  in theorem 5.1.1, then resulting inequality for S-convex FIVF as follows

$$\begin{aligned} & 2^{s-1} \mathfrak{S} \left( \frac{\bar{\theta} + \rho}{2} \right) \leq \frac{\Gamma(\mathfrak{J}+1)}{2(\rho-\bar{\theta})^{\mathfrak{J}}} [\Lambda_{\bar{\theta}}^{\mathfrak{J}} \mathfrak{S}(\rho) \tilde{+} \Lambda_{\rho}^{\mathfrak{J}} \mathfrak{S}(\bar{\theta})] \leq \frac{\mathfrak{S}(\bar{\theta}) \tilde{+} \mathfrak{S}(\rho)}{2} \times \\ & \int_0^1 [\theta^s + (1-\theta)^s] \theta^{\mathfrak{J}-1} d\theta. \end{aligned}$$

2. If we take the value  $\mathfrak{J} = 1$  in theorem 5.1.1, we have the following inequality

$$2^{s-1} \mathfrak{S} \left( \frac{\bar{\theta} + \rho}{2} \right) \leq \frac{1}{\rho-\bar{\theta}} \int_{\bar{\theta}}^{\rho} \mathfrak{S}(\theta) d\theta \leq \frac{\mathfrak{S}(\bar{\theta}) \tilde{+} \mathfrak{S}(\rho)}{s+1}.$$

## 4.2 Modification of the Trapezoid Type Inequalities for h-Godunova- Levin Pre-Invex FIVF

Here, we develop the lemma for h-Godunova-Levin pre-invex FIVF, which helpful to proved our main results. We develop the trapezoid type inequalities for h-Godunova- Levin pre-invex FIVF, and deduced some well known inequalities from our main results, are discussed in the form of corollaries.

**Lemma 4.2.1.** Let  $\mathfrak{S} : \Theta = [\vartheta, \vartheta + \xi(\varrho, \vartheta)] \rightarrow \mathfrak{R}$  be real valued and differentiable function, and  $\Theta$  be an open-invex set with respect to  $\xi : \Theta \times \Theta \rightarrow \mathfrak{R}$ ,  $\xi(\varrho, \vartheta) > 0$  for  $\vartheta, \varrho \in \Theta$ , then the following inequality holds:

$$\frac{\mathfrak{S}(\vartheta) + \mathfrak{S}(\vartheta + \xi(\varrho, \vartheta))}{2} - \frac{\Gamma(\mathfrak{J}+1)}{2\xi^{\mathfrak{J}+1}(\varrho, \vartheta)} [\Lambda_{\vartheta+}^{\mathfrak{J}} \mathfrak{S}(\vartheta + \xi(\varrho, \vartheta)) + \Lambda_{\vartheta+\xi(\varrho, \vartheta)}^{\mathfrak{J}} \mathfrak{S}(\vartheta)] \\ = \frac{\xi(\varrho, \vartheta)}{2} \int_0^1 [\theta^{\mathfrak{J}} - (1-\theta)^{\mathfrak{J}}] \mathfrak{S}'(\vartheta + \theta \xi(\varrho, \vartheta)) d\theta.$$

*Proof.* Consider the integral

$$\mathfrak{I} = \int_0^1 [\theta^{\mathfrak{J}} - (1-\theta)^{\mathfrak{J}}] \mathfrak{S}'(\vartheta + \theta \xi(\varrho, \vartheta)) d\theta$$

If we take for every  $\theta \in [0, 1]$ , we have

$$= \int_0^1 [\theta^{\mathfrak{J}} - (1-\theta)^{\mathfrak{J}}] \mathfrak{S}'_*(\vartheta + \theta \xi(\varrho, \vartheta)) d\theta \\ = \int_0^1 \theta^{\mathfrak{J}} \mathfrak{S}'_*(\vartheta + \theta \xi(\varrho, \vartheta)) d\theta - \int_0^1 (1-\theta)^{\mathfrak{J}} \mathfrak{S}'_*(\vartheta + \theta \xi(\varrho, \vartheta)) d\theta \quad (4.15)$$

Now solve the integrals of equation (5.14), by considering as  $\mathfrak{I}_1$  and  $\mathfrak{I}_2$

For evaluating  $\mathfrak{I}_1$

$$= \theta^{\mathfrak{J}} \frac{\mathfrak{S}_*(\vartheta + \theta \xi(\varrho, \vartheta))}{\xi(\alpha_2, \alpha_1)} \Big|_0^1 - \int_0^1 \mathfrak{J} \theta^{\mathfrak{J}-1} \frac{\mathfrak{S}_*(\vartheta + \theta \xi(\varrho, \vartheta))}{\xi(\alpha_2, \alpha_1)} d\theta$$

$$\mathfrak{I}_1 = \frac{\mathfrak{S}_*(\vartheta + \xi(\varrho, \vartheta))}{\xi(\varrho, \vartheta)} - \frac{\mathfrak{J}}{\xi(\varrho, \vartheta)} \int_0^1 \theta^{\mathfrak{J}-1} \mathfrak{S}_*(\vartheta + \theta \xi(\varrho, \vartheta)) d\theta$$

$$\mathfrak{I}_1 = \frac{\mathfrak{S}_*(\vartheta + \xi(\varrho, \vartheta))}{\xi(\varrho, \vartheta)} - \frac{\Gamma(\mathfrak{J}+1)}{\xi^{\mathfrak{J}+1}(\varrho, \vartheta)} [\mathfrak{J}_{(\vartheta+\xi(\varrho, \vartheta))}^{\mathfrak{J}} \mathfrak{S}_*(\vartheta)]$$

Continue with same manner to solve  $\mathfrak{I}_2$ ,

$$\mathfrak{I}_2 = \frac{\mathfrak{S}_*(\vartheta)}{\xi(\varrho, \vartheta)} - \frac{\Gamma(\mathfrak{J}+1)}{\xi^{\mathfrak{J}+1}(\varrho, \vartheta)} [\Lambda_{\vartheta+}^{\mathfrak{J}} \mathfrak{S}_*(\vartheta + \xi(\varrho, \vartheta))]$$

$$\frac{\mathfrak{S}_*(\vartheta) + \mathfrak{S}_*(\vartheta + \xi(\varrho, \vartheta))}{2} - \frac{\Gamma(\mathfrak{J}+1)}{2\xi^{\mathfrak{J}+1}(\varrho, \vartheta)} [\Lambda_{(\vartheta+\xi(\varrho, \vartheta))}^{\mathfrak{J}} \mathfrak{S}_*(\vartheta) + \Lambda_{\vartheta+}^{\mathfrak{J}} \mathfrak{S}_*(\vartheta + \xi(\varrho, \vartheta))] \\ = \frac{\xi^{\mathfrak{J}}(\varrho, \vartheta)}{2} \int_0^1 [\theta^{\mathfrak{J}} - (1-\theta)^{\mathfrak{J}}] \mathfrak{S}'_*(\vartheta + \theta \xi(\varrho, \vartheta)) d\theta \quad (4.16)$$

Similarly we solve for upper FIVF  $\mathfrak{S}^*$ , we obtain

$$\frac{\mathfrak{S}^*(\vartheta) + \mathfrak{S}^*(\vartheta + \xi(\varrho, \vartheta))}{2} - \frac{\Gamma(\mathfrak{J}+1)}{2\xi^{\mathfrak{J}+1}(\varrho, \vartheta)} [\Lambda_{(\vartheta+\xi(\varrho, \vartheta))}^{\mathfrak{J}} \mathfrak{S}^*(\vartheta) + \Lambda_{\vartheta+}^{\mathfrak{J}} \mathfrak{S}^*(\vartheta + \xi(\varrho, \vartheta))] \\ = \frac{\xi^{\mathfrak{J}}(\varrho, \vartheta)}{2} \int_0^1 [\theta^{\mathfrak{J}} - (1-\theta)^{\mathfrak{J}}] \mathfrak{S}'^*(\vartheta + \theta \xi(\varrho, \vartheta)) d\theta \quad (4.17)$$

By combining equation (5.18) and (5.19), the we have

$$\frac{\mathfrak{S}_*(\vartheta) + \mathfrak{S}_*(\vartheta + \xi(\varrho, \vartheta)) + \mathfrak{S}^*(\vartheta) + \mathfrak{S}^*(\vartheta + \xi(\varrho, \vartheta))}{2} - \frac{\Gamma(\mathfrak{J}+1)}{2\xi^{\mathfrak{J}+1}(\varrho, \vartheta)} \\ [\Lambda_{(\vartheta+\xi(\varrho, \vartheta))}^{\mathfrak{J}} \mathfrak{S}_*(\vartheta) + \Lambda_{\vartheta+}^{\mathfrak{J}} \mathfrak{S}_*(\vartheta + \xi(\varrho, \vartheta)) + \Lambda_{(\vartheta+\xi(\varrho, \vartheta))}^{\mathfrak{J}} \mathfrak{S}^*(\vartheta) + \Lambda_{\vartheta+}^{\mathfrak{J}} \mathfrak{S}^*(\vartheta + \xi(\varrho, \vartheta))] \\ = \frac{\xi^{\mathfrak{J}}(\varrho, \vartheta)}{2} \int_0^1 [\theta^{\mathfrak{J}} - (1-\theta)^{\mathfrak{J}}] (\mathfrak{S}'_*(\vartheta + \theta \xi(\varrho, \vartheta)) + \mathfrak{S}'^*(\vartheta + \theta \xi(\varrho, \vartheta))) d\theta$$

That is,

$$\frac{\mathfrak{S}(\vartheta) + \mathfrak{S}(\vartheta + \xi(\varrho, \vartheta))}{2} - \frac{\Gamma(\mathfrak{J}+1)}{2\xi^{\mathfrak{J}+1}(\varrho, \vartheta)} [\Lambda_{\vartheta+}^{\mathfrak{J}} \mathfrak{S}(\vartheta + \xi(\varrho, \vartheta)) + \Lambda_{(\vartheta+\xi(\varrho, \vartheta))}^{\mathfrak{J}} \mathfrak{S}(\vartheta)] \\ = \frac{\xi(\varrho, \vartheta)}{2} \int_0^1 [\theta^{\mathfrak{J}} - (1-\theta)^{\mathfrak{J}}] \mathfrak{S}'(\vartheta + \theta \xi(\varrho, \vartheta)) d\theta.$$



From the Lemma 5.2.1, we have the following theorem.

**Theorem 4.2.1.** Let  $\mathfrak{Z} : \Theta = [\vartheta, \vartheta + \xi(\varrho, \vartheta)] \rightarrow (0, \infty)$  real valued function and  $|\mathfrak{Z}'|$  be differentiable  $h$ -Godunova-Levin pre-invex FIVF on  $\Theta$ , then the following fuzzy fractional inequality holds:

$$\left| \frac{\mathfrak{Z}(\vartheta) \tilde{+} \mathfrak{Z}(\vartheta + \xi(\varrho, \vartheta))}{2} - \frac{\Gamma(\mathfrak{I} + 1)}{2\xi^{\mathfrak{I}}(\varrho, \vartheta)} \left[ \Lambda_{\vartheta}^{\mathfrak{I}} \mathfrak{Z}(\vartheta + \xi(\varrho, \vartheta)) \tilde{+} \Lambda_{\vartheta + \xi(\varrho, \vartheta)}^{\mathfrak{I}} \mathfrak{Z}(\vartheta) \right] \right| \leq \frac{\xi(\varrho, \vartheta)}{2} [|\mathfrak{Z}'(\vartheta)| \tilde{+} |\mathfrak{Z}'(\varrho)|] \int_0^1 |\theta^{\mathfrak{I}} - (1 - \theta)^{\mathfrak{I}}| \frac{1}{h(\theta)} d\theta.$$

*Proof.*

$$\left| \frac{\mathfrak{Z}(\vartheta) \tilde{+} \mathfrak{Z}(\vartheta + \xi(\varrho, \vartheta))}{2} - \frac{\Gamma(\mathfrak{I} + 1)}{2\xi^{\mathfrak{I}}(\varrho, \vartheta)} \left[ \Lambda_{\vartheta}^{\mathfrak{I}} \mathfrak{Z}(\vartheta + \xi(\varrho, \vartheta)) \tilde{+} \Lambda_{\vartheta + \xi(\varrho, \vartheta)}^{\mathfrak{I}} \mathfrak{Z}(\vartheta) \right] \right| = \left| \frac{\xi(\varrho, \vartheta)}{2} \int_0^1 [\theta^{\mathfrak{I}} - (1 - \theta)^{\mathfrak{I}}] \mathfrak{Z}'(\vartheta + \theta \xi(\varrho, \vartheta)) d\theta \right|$$

If we take for every  $\theta \in [0, 1]$ , we have

$$\begin{aligned} &= \left| \frac{\xi(\varrho, \vartheta)}{2} \int_0^1 [\theta^{\mathfrak{I}} - (1 - \theta)^{\mathfrak{I}}] \mathfrak{Z}'(\vartheta + \theta \xi(\varrho, \vartheta)) d\theta \right| \\ &\leq \frac{\xi(\varrho, \vartheta)}{2} \int_0^1 |\theta^{\mathfrak{I}} - (1 - \theta)^{\mathfrak{I}}| |\mathfrak{Z}'(\vartheta + \theta \xi(\varrho, \vartheta))| d\theta \end{aligned}$$

By applying definition of  $h$ -Godunova-Levin pre-invex FIVF on  $|\mathfrak{Z}'(\vartheta + \theta \xi(\varrho, \vartheta))|$ , we have

$$\begin{aligned} &\leq \frac{\xi(\varrho, \vartheta)}{2} \int_0^1 |\theta^{\mathfrak{I}} - (1 - \theta)^{\mathfrak{I}}| \left[ \frac{|\mathfrak{Z}'(\vartheta)|}{h(\theta)} + \frac{|\mathfrak{Z}'(\varrho)|}{h(1 - \theta)} \right] d\theta \\ &\leq \frac{\xi(\varrho, \vartheta)}{2} \int_0^1 |\theta^{\mathfrak{I}} - (1 - \theta)^{\mathfrak{I}}| \left[ \frac{|\mathfrak{Z}'(\vartheta)|}{h(\theta)} + \frac{|\mathfrak{Z}'(\varrho)|}{h(1 - \theta)} \right] d\theta \end{aligned} \quad (4.18)$$

similarly, we solve for upper FIVF  $\mathfrak{Z}^*$ , we obtain,

$$\leq \frac{\xi(\varrho, \vartheta)}{2} \int_0^1 |\theta^{\mathfrak{I}} - (1 - \theta)^{\mathfrak{I}}| \left[ \frac{|\mathfrak{Z}'^*(\vartheta)|}{h(\theta)} + \frac{|\mathfrak{Z}'^*(\varrho)|}{h(1 - \theta)} \right] d\theta \quad (4.19)$$

from equation (5.21) and (5.23)

$$\begin{aligned} &\leq \frac{\xi(\varrho, \vartheta)}{2} [|\mathfrak{Z}'(\vartheta)| + |\mathfrak{Z}'(\varrho)|, |\mathfrak{Z}'^*(\vartheta)| + |\mathfrak{Z}'^*(\varrho)|] \int_0^1 |\theta^{\mathfrak{I}} - (1 - \theta)^{\mathfrak{I}}| \frac{1}{h(\theta)} d\theta, \\ &\leq \frac{\xi(\varrho, \vartheta)}{2} [|\mathfrak{Z}'(\vartheta)| \tilde{+} |\mathfrak{Z}'(\varrho)|] \int_0^1 |\theta^{\mathfrak{I}} - (1 - \theta)^{\mathfrak{I}}| \frac{1}{h(\theta)} d\theta, \end{aligned}$$

as required.  $\square$

**Corollary 4.2.1.** If we substitute the value of  $\mathfrak{I} = 1$  in the theorem 5.2.1, then we have the following inequality holds:

$$\left| \frac{\mathfrak{Z}(\vartheta) \tilde{+} \mathfrak{Z}(\vartheta + \xi(\varrho, \vartheta))}{2} - \frac{1}{\xi(\varrho, \vartheta)} \int_{\vartheta}^{\vartheta + \xi(\varrho, \vartheta)} \mathfrak{Z}(\theta) d\theta \right| \leq \frac{\xi(\varrho, \vartheta)}{2} [|\mathfrak{Z}'(\vartheta)| \tilde{+} |\mathfrak{Z}'(\varrho)|] \int_0^1 \frac{|1 - 2\theta|}{h(\theta)} d\theta.$$

**Corollary 4.2.2.** By putting the values  $\xi(\varrho, \vartheta) = \varrho - \vartheta$  in the theorem 5.2.1, we obtain the following inequality

$$\left| \frac{\mathfrak{Z}(\vartheta) \tilde{+} \mathfrak{Z}(\varrho)}{2} - \frac{\Gamma(\mathfrak{I} + 1)}{2(\varrho - \vartheta)^{\mathfrak{I}}} \left[ \Lambda_{\vartheta}^{\mathfrak{I}} \mathfrak{Z}(\varrho) \tilde{+} \Lambda_{\varrho}^{\mathfrak{I}} \mathfrak{Z}(\vartheta) \right] \right| \leq \frac{\varrho - \vartheta}{2} [|\mathfrak{Z}'(\vartheta)| \tilde{+} |\mathfrak{Z}'(\varrho)|] \int_0^1 |\theta^{\mathfrak{I}} - (1 - \theta)^{\mathfrak{I}}| \frac{1}{h(\theta)} d\theta.$$

**Corollary 4.2.3.** 1. By replacing  $\mathfrak{I} = 1$  in theorem ??, we get:

$$\left| \frac{\mathfrak{Z}(\vartheta) \tilde{+} \mathfrak{Z}(\vartheta + \xi(\varrho, \vartheta))}{2} - \frac{1}{\xi(\varrho, \vartheta)} \int_{\vartheta}^{\vartheta + \xi(\varrho, \vartheta)} \mathfrak{Z}(\theta) d\theta \right| \leq \frac{\xi(\varrho, \vartheta)}{2(p+1)^{\frac{1}{p}}} (|\mathfrak{Z}'(\vartheta)|^q \tilde{+} |\mathfrak{Z}'(\varrho)|^q)^{\frac{1}{q}} \left( \int_0^1 \frac{1}{h(\theta)} d\theta \right)^{\frac{1}{q}}.$$

2. Now putting  $\xi(\rho, \delta) = \rho - \delta$  and  $h(\theta) = \frac{1}{\theta^s}$ , we obtain:

$$\left| \frac{\mathfrak{Z}(\delta) + \mathfrak{Z}(\rho)}{2} - \frac{1}{(\rho - \delta)} \int_{\delta}^{\rho} \mathfrak{Z}(\theta) d\theta \right| \leq \frac{(\rho - \delta)}{2(p+1)^{\frac{1}{p}}} \left( \frac{|\mathfrak{Z}'(\delta)|^q + |\mathfrak{Z}'(\rho)|^q}{s+1} \right)^{\frac{1}{q}}.$$

## Examine the Behavior of Hermite-Hadamard (H-H) Type Inequalities for h-Godunova-Levin Convex Fuzzy-IVF

In this section, we discuss the behavior of Hermite-Hadamard type inequalities for h-Godunova-Levin convex fuzzy IVF with the help of E1vBMF and deduce the wellknown inequalities from our main result, which are discuss in the form of corollaries.

**Theorem 5.1.1.** Let  $h : (0, 1) \rightarrow \mathfrak{R}$  be a positive real valued function and  $\mathfrak{Z} \in L_1[\delta, \rho]$ , such that  $h(\theta) \neq 0$ ,  $\mathfrak{Z} : [\delta, \rho] \rightarrow \mathfrak{R}$  be h-Godunova-Levin convex fuzzy IVF, where  $0 < \delta < \rho$ , then the following fuzzy fractional integral inequality holds

$$\begin{aligned} & \frac{h(\frac{1}{2})}{2} \mathfrak{Z}\left(\frac{\delta + \rho}{2}\right) \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \delta_1}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \right)(x, r) \\ & \leq \frac{1}{2} \left[ \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \delta_1}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \mathfrak{Z} \right)(x, r) + \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \delta_1}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \mathfrak{Z} \right)(x, r) \right] \\ & \leq \frac{(\mathfrak{Z}(\delta) + \mathfrak{Z}(\rho))}{2} \int_0^1 \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \theta^{\phi'} \mathbb{J}_{\phi', r, q, \sigma, m, \rho}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} (\omega \theta^{\alpha_1; p}) d\theta. \end{aligned}$$

Where  $\omega' = \frac{\omega}{\xi(\delta, \rho)^{\alpha_1}}$

*Proof.* Consider the definition of h-Godunova-Levin convex function over the closed interval  $[\delta, \rho]$ , and  $x, y \in [\delta, \rho]$ , we have

$$\mathfrak{Z}(\delta x + (1 - \delta)y) \leq \frac{\mathfrak{Z}(x)}{h(\delta)} + \frac{\mathfrak{Z}(y)}{h(1 - \delta)} \quad (5.1)$$

Putting the values  $x = \theta\bar{\delta} + (1-\theta)\rho, y = (1-\theta)\bar{\delta} + \theta\rho$  and  $\delta = \frac{1}{2}$ , we have

$$h\left(\frac{1}{2}\right) \mathfrak{S}\left(\frac{\bar{\delta} + \rho}{2}\right) \leq [\mathfrak{S}(\theta\bar{\delta} + (1-\theta)\rho) + \mathfrak{S}((1-\theta)\bar{\delta} + \theta\rho)]. \quad (5.2)$$

If we take for every  $\theta \in [0, 1]$ , we have

$$h\left(\frac{1}{2}\right) \mathfrak{S}_*\left(\frac{\bar{\delta} + \rho}{2}\right) \leq [\mathfrak{S}_*(\theta\bar{\delta} + (1-\theta)\rho) + \mathfrak{S}_*((1-\theta)\bar{\delta} + \theta\rho)]. \quad (5.3)$$

Multiplying  $\theta^{\phi'} \mathbb{J}_{\phi', r, q, \sigma, m, \rho}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau}(\omega\theta^{\alpha_1}; p)$  in the equation (5.3) and then integrating with respect to  $\theta$  over the interval  $[0, 1]$ , we obtain

$$\begin{aligned} & h\left(\frac{1}{2}\right) \mathfrak{S}_*\left(\frac{\bar{\delta} + \rho}{2}\right) \int_0^1 \theta^{\phi'} \mathbb{J}_{\phi', r, q, \sigma, m, \rho}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau}(\omega\theta^{\alpha_1}; p) d\theta \\ & \leq \int_0^1 \theta^{\phi'} \mathbb{J}_{\phi', r, q, \sigma, m, \rho}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau}(\omega\theta^{\alpha_1}; p) \mathfrak{S}_*(\theta\bar{\delta} + (1-\theta)\rho) d\theta + \\ & \int_0^1 \theta^{\phi'} \mathbb{J}_{\phi', r, q, \sigma, m, \rho}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau}(\omega\theta^{\alpha_1}; p) \mathfrak{S}_*((1-\theta)\bar{\delta} + \theta\rho) d\theta. \\ & h\left(\frac{1}{2}\right) \mathfrak{S}_*\left(\frac{\bar{\delta} + \rho}{2}\right) \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + \eta_{1n}, \tau - \gamma)(\tau) \eta_{1n}(\sigma) \nu_{1n}(\rho_1)_{mn} (-\omega)^n}{\beta(\gamma, \tau - \gamma) \Gamma(\alpha_{1n} + \phi' + 1)(q) \xi_{1n}(\delta_1)_{pn}} \times \\ & \int_0^1 \theta^{\phi' + \alpha_{1n}} d\theta \leq \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + \eta_{1n}, \tau - \gamma)(\tau) \eta_{1n}(\sigma) \nu_{1n}(\rho_1)_{mn} (-\omega)^n}{\beta(\gamma, \tau - \gamma) \Gamma(\alpha_{1n} + \phi' + 1)(q) \xi_{1n}(\delta_1)_{pn}} \times \\ & \left[ \int_0^1 \theta^{\phi' + \alpha_{1n}} \mathfrak{S}_*(\theta\bar{\delta} + (1-\theta)\rho) d\theta + \int_0^1 \theta^{\phi' + \alpha_{1n}} \mathfrak{S}_*((1-\theta)\bar{\delta} + \theta\rho) d\theta \right]. \end{aligned} \quad (5.4)$$

By solving the integrals involve in (5.4), we get;

$$\frac{h(\frac{1}{2})}{2} \mathfrak{S}_*\left(\frac{\bar{\delta} + \rho}{2}\right) \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \delta_1}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \right)(x, r) \leq \frac{1}{2} \left[ \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \delta_1}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \right)(x, r) + \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \delta_1}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \right)(x, r) \right]. \quad (5.5)$$

Now, we consider  $h$ -Godunova-Levin convexity of  $\mathfrak{S}$ , for second part of inequality,

$$\mathfrak{S}(\theta\bar{\delta} + (1-\theta)\rho) \leq \frac{\mathfrak{S}(\bar{\delta})}{h(\theta)} + \frac{\mathfrak{S}(\rho)}{h(1-\theta)}, \quad (5.6)$$

and

$$\mathfrak{S}((1-\theta)\bar{\delta} + \theta\rho) \leq \frac{\mathfrak{S}(\bar{\delta})}{h(1-\theta)} + \frac{\mathfrak{S}(\rho)}{h(\theta)}. \quad (5.7)$$

By adding equation (5.6) and (5.7), we get

$$\mathfrak{S}(\theta\bar{\delta} + (1-\theta)\rho) + \mathfrak{S}((1-\theta)\bar{\delta} + \theta\rho) \leq (\mathfrak{S}(\bar{\delta}) + \mathfrak{S}(\rho)) \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \quad (5.8)$$

If we take for every  $\theta \in [0, 1]$ , we have

$$\mathfrak{S}_*(\theta\bar{\delta} + (1-\theta)\rho) + \mathfrak{S}_*((1-\theta)\bar{\delta} + \theta\rho) \leq (\mathfrak{S}_*(\bar{\delta}) + \mathfrak{S}_*(\rho)) \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right]. \quad (5.9)$$

By multiplying  $\theta^{\phi'} \mathbb{J}_{\phi', r, q, \sigma, m, \rho}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau}(\omega\theta^{\alpha_1}; p)$  on both sides of equation (5.9) and integrating inequality over interval  $[0, 1]$  with respect to  $\theta$ , we get

$$\begin{aligned} & \int_0^1 \theta^{\phi'} \mathbb{J}_{\phi', r, q, \sigma, m, \rho}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau}(\omega\theta^{\alpha_1}; p) \mathfrak{S}_*(\theta\bar{\delta} + (1-\theta)\rho) d\theta \\ & + \int_0^1 \theta^{\phi'} \mathbb{J}_{\phi', r, q, \sigma, m, \rho}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau}(\omega\theta^{\alpha_1}; p) \mathfrak{S}_*((1-\theta)\bar{\delta} + \theta\rho) d\theta \\ & \leq (\mathfrak{S}_*(\bar{\delta}) + \mathfrak{S}_*(\rho)) \int_0^1 \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \theta^{\phi'} \mathbb{J}_{\phi', r, q, \sigma, m, \rho}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau}(\omega\theta^{\alpha_1}; p) d\theta. \end{aligned}$$



After simplification, we obtain

$$\begin{aligned} & \frac{1}{2} \left[ \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^+}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \mathfrak{Z}_* \right) (x, r) + \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^-}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \mathfrak{Z}_* \right) (x, r) \right] \\ & \leq \frac{\left( \mathfrak{Z}_*(\bar{\theta}) + \mathfrak{Z}_*(\beta) \right)}{2} \int_0^1 \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \times \\ & \theta^{\phi'} \mathbb{J}_{\phi', r, q, \sigma, m, \rho}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} (\omega \theta^{\alpha_1}; p) d\theta. \end{aligned} \quad (5.10)$$

By combining (5.5) and (5.10), we get:

$$\begin{aligned} & \frac{h(\frac{1}{2})}{2} \mathfrak{Z}_* \left( \frac{\bar{\theta} + \beta}{2} \right) \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^+}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \right) (x, r) \leq \frac{1}{2} \left[ \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^+}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \mathfrak{Z}_* \right) (x, r) + \right. \\ & \left. \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^-}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \mathfrak{Z}_* \right) (x, r) \right] \leq \frac{\left( \mathfrak{Z}_*(\bar{\theta}) + \mathfrak{Z}_*(\beta) \right)}{2} \int_0^1 \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \times \\ & \theta^{\phi'} \mathbb{J}_{\phi', r, q, \sigma, m, \rho}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} (\omega \theta^{\alpha_1}; p) d\theta. \end{aligned} \quad (5.11)$$

Similarly, we solve the upper fuzzy IVF for  $\mathfrak{Z}^*$ , we obtain

$$\begin{aligned} & \frac{h(\frac{1}{2})}{2} \mathfrak{Z}^* \left( \frac{\bar{\theta} + \beta}{2} \right) \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^-}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \right) (x, r) \\ & \leq \frac{1}{2} \left[ \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^+}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \mathfrak{Z}^* \right) (x, r) + \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^-}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \mathfrak{Z}^* \right) (x, r) \right] \\ & \leq \frac{\left( \mathfrak{Z}^*(\bar{\theta}) + \mathfrak{Z}^*(\beta) \right)}{2} \int_0^1 \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \times \\ & \theta^{\phi'} \mathbb{J}_{\phi', r, q, \sigma, m, \rho}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} (\omega \theta^{\alpha_1}; p) d\theta. \end{aligned} \quad (5.12)$$

Combining equation (5.11) and (5.12), we get the following inequality.

$$\begin{aligned} & \frac{h(\frac{1}{2})}{2} \left[ \mathfrak{Z}_* \left( \frac{\bar{\theta} + \beta}{2} \right), \mathfrak{Z}^* \left( \frac{\bar{\theta} + \beta}{2} \right) \right] \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^+}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \right) (x, r) \leq \\ & \frac{1}{2} \left[ \left\{ \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^+}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \mathfrak{Z}_* \right) (x, r) + \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^-}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \mathfrak{Z}_* \right) (x, r) \right\}, \right. \\ & \left. \left\{ \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^+}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \mathfrak{Z}^* \right) (x, r) + \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^-}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \mathfrak{Z}^* \right) (x, r) \right\} \right] \\ & \leq \frac{\left( \mathfrak{Z}_*(\bar{\theta}), \mathfrak{Z}^*(\bar{\theta}) + \mathfrak{Z}_*(\beta), \mathfrak{Z}^*(\beta) \right)}{2} \times \\ & \int_0^1 \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \theta^{\phi'} \mathbb{J}_{\phi', r, q, \sigma, m, \rho}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} (\omega \theta^{\alpha_1}; p) d\theta. \end{aligned}$$

That is

$$\begin{aligned} & \frac{h(\frac{1}{2})}{2} \mathfrak{Z} \left( \frac{\bar{\theta} + \beta}{2} \right) \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^+}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \right) (x, r) \\ & \leq \frac{1}{2} \left[ \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^+}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \mathfrak{Z} \right) (x, r) + \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^-}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \mathfrak{Z} \right) (x, r) \right] \\ & \leq \frac{\left( \mathfrak{Z}(\bar{\theta}) + \mathfrak{Z}(\beta) \right)}{2} \int_0^1 \left[ \frac{1}{h(\theta)} + \frac{1}{h(1-\theta)} \right] \times \\ & \theta^{\phi'} \mathbb{J}_{\phi', r, q, \sigma, m, \rho}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} (\omega \theta^{\alpha_1}; p) d\theta. \end{aligned} \quad (5.13)$$

which is our require result.  $\square$

**Corollary 5.1.1.** By putting the value  $h(\theta) = \theta^s$  in theorem 5.1.1, then we obtain the (H – H)-type inequality for s-Godunova-Levin convex fuzzy IVF, as given below

$$\begin{aligned} & \frac{(\frac{1}{2})^s}{2} \mathfrak{Z} \left( \frac{\bar{\theta} + \beta}{2} \right) \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^+}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \right) (x, r) \leq \frac{1}{2} \left[ \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^+}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \mathfrak{Z} \right) (x, r) + \right. \\ & \left. \left( \mathbb{E}_{\phi', r, q, \sigma, m, \rho, \beta^-}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} \mathfrak{Z} \right) (x, r) \right] \leq \frac{\mathfrak{Z}(\bar{\theta}) + \mathfrak{Z}(\beta)}{2} \int_0^1 \left[ \frac{1}{(\theta)^s} + \frac{1}{(1-\theta)^s} \right] \times \\ & \theta^{\phi'} \mathbb{J}_{\phi', r, q, \sigma, m, \rho}^{\alpha_1, \eta_1, \xi_1, \nu_1, \rho_1, \delta_1, \tau} (\omega \theta^{\alpha_1}; p) d\theta. \end{aligned}$$

**Corollary 5.1.2.** If we take the value  $h(\theta) = \theta$  in theorem 5.1.1, then obtain the integral



## Discussion & Conclusion

In this study presents a comprehensive and methodical extension of trapezoid-type inequalities through the mathematical framework of  $h$ -Godunova–Levin pre-invex fuzzy interval-valued functions (FIVFs) under Riemann–Liouville fractional integral operators. This section establishes a solid theoretical foundation for analyzing fuzzy fractional inequalities by combining concepts from convexity, pre-invexity, and fractional calculus—three of the most active and interlinked domains in modern mathematical analysis. The approach taken in this study not only generalizes classical inequalities but also enhances their analytical scope by embedding fuzzy logic and fractional differentiation, enabling a more accurate representation of uncertain or imprecise systems. The formulation begins with Lemma 5.2.1, which serves as the cornerstone for the subsequent theorems. This lemma formalizes a critical relationship between differentiable fuzzy IVFs and their corresponding fractional operators, thereby linking the behavior of fuzzy functions with fractional integral forms. The lemma paves the way for defining new integral bounds that capture both the non-linearity of pre-invex structures and the memory-dependent nature of fractional operators. Such an integration is crucial in extending classical deterministic inequalities into fuzzy fractional environments, where uncertainty and non-local effects are significant. Following this foundation, Theorem 5.2.1 introduces a generalized fuzzy fractional trapezoid-type inequality based on the  $h$ -Godunova–Levin convexity principle. This theorem not only broadens the scope of existing inequalities but also provides a new analytical lens to examine functions that do not necessarily adhere to traditional convexity assumptions. By introducing the  $h$ -function as a control parameter, the study captures a continuum between convex and non-convex behaviors, thereby enhancing the flexibility and applicability of the derived inequalities. This result is particularly valuable because it accommodates a wide class of fuzzy interval-valued mappings, extending the theoretical framework beyond crisp real-valued functions.

The Corollaries 5.2.1 and 5.2.2 derived from the theorem demonstrate the adaptability and generality of the presented inequalities. By choosing specific forms of the  $h$ -function and fractional order, the general inequality simplifies into well-known classical results such as the Hermite–Hadamard, trapezoid, and Simpson-type

inequalities. These corollaries validate the theoretical soundness of the proposed generalization, confirming that the introduced models are consistent with and inclusive of previously established results. They also highlight how fractional operators, when integrated with fuzzy structures, can yield more nuanced and realistic estimates of inequalities in the presence of uncertainty. Theorem 5.2.2 extends the Hermite–Hadamard inequality to the fuzzy fractional setting, emphasizing the interconnectedness between invexity theory and fractional calculus within fuzzy environments. The theorem illustrates that by relaxing the conventional convexity constraints, one can derive more general bounds applicable to functions with broader structural variability. This advancement demonstrates how fuzzy fractional approaches can offer superior flexibility in mathematical modeling, particularly in systems where both uncertainty and hereditary characteristics play a critical role—such as in control theory, signal processing, and decision analysis. The results presented in this section signify a substantial advancement in the study of fuzzy fractional inequalities. The integration of  $h$ -Godunova–Levin pre-invexity with Riemann–Liouville fractional operators introduces a new analytical paradigm that bridges deterministic and fuzzy fractional domains. The findings not only enrich the theoretical understanding of inequalities under fuzziness but also pave the way for practical applications in fields such as optimization, computational intelligence, engineering mathematics, and applied physics. In these domains, systems often exhibit both fractional dynamics—arising from memory and hereditary effects—and fuzzy uncertainty, stemming from imprecise measurements or linguistic variables.

Thus, the mathematical framework developed in this section provides a robust, unified structure for investigating inequalities under combined fuzzy and fractional conditions. By extending classical inequalities through the inclusion of  $h$ -convexity and pre-invexity, the research achieves a deeper level of generalization and adaptability. This theoretical contribution not only solidifies the role of fractional fuzzy calculus in modern analysis but also opens new research directions for constructing refined inequalities, developing numerical approximations, and exploring applications in multi-criteria optimization and decision-making under uncertainty. In essence, Section 5.2 marks an important contribution to the

evolution of inequality theory, reflecting both mathematical depth and interdisciplinary applicability.

In this research thesis, we explored the development of well-known inequalities for two dimensional  $h$ -Godunova-Levin convex functions and coordinated  $h$ -Godunova-Levin pre-invex functions. We presented modified forms of Hermite-Hadamard type fractional integral inequalities and trapezoid-type inequalities tailored to these functions. The extensions and generalizations of convexity and non-convexity (pre-invexity) through fuzzy interval-valued functions (FIVFs) represent a significant contribution to the study of inequalities. In this work, we introduced a new class of convex and non-convex FIVFs constructed using arbitrary auxiliary  $h$ -functions. This framework enabled us to derive new versions of Hermite-Hadamard and trapezoid-type inequalities. In future research, we aim to further investigate this approach for generalized convex fuzzy interval-valued functions and their applications in fuzzy interval nonlinear programming. Applying these ideas is expected to open new directions in convex analysis and optimization theory.

on theory.

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