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# Initial and Logarithmic Coefficient Bounds for Functions Associated with the Lemniscate of Bernoulli

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## ABSTRACT

This paper studies coefficient problems for functions starlike with respect to symmetric points associated with the lemniscate of Bernoulli, defined by a subordination condition involving
$\sqrt{1+z}$. Using the Schwarz-function method, we obtain first three bounds for the initial Taylor coefficients and the first three logarithmic coefficients. These results extend coefficient-problem techniques to this class of analytic functions.

**Keywords:** Bernoulli Lemniscate; Symmetric Points, Taylor Coefficients; Logarithmic Coefficients; Subordination.

## Introduction:

Geometric Function Theory is an important branch of complex analysis concerned with the geometric properties of analytic and univalent functions in the open unit disk. One of its fundamental research areas is the study of coefficient estimates for normalized analytic functions. These estimates provide information about the behavior of analytic functions and play an important role in investigating coefficient functional, logarithmic coefficients, and Hankel determinants. In particular, coefficient problems for subclasses of starlike functions have received considerable attention because of their connections with geometric properties and subordination theory.

Let $U = \{z: |z| < 1\}$ denote the open unit disk, and let $A$ represent the family of normalized analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (1.1)$$

satisfying the normalization conditions $f(0) = f'(0) - 1 = 0$.

A function is said to be starlike with respect to symmetric points if it satisfies the

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geometric condition

$$Re\left\{\frac{2zf'(z)}{(z)-f(-z)}\right\} > 0,\ z\epsilon U$$

whenever the quotient is analytic. This class is a natural extension of the classical theory of starlike functions and provides a framework for investigating coefficient problems involving the values of a function at symmetric points.

The coefficient problem consists of determining the best possible bounds for the Taylor coefficients $a_K$ and related coefficient functionals associated with various subclasses of
$A$.Subordination theory provides one of the most effective techniques for studying coefficient problems. For analytic functions $f$ and $g$ in $U$, we say that $f$ is subordinate to $g$, written as $f \prec$
$g$, if there exists a Schwarz function

$$w(z) = \sum_{k=1}^{\infty} c_K z^K$$

, $|(z)| < 1, \qquad w(0) = 0.$ (1.2)

such that $f(z) = g(w(z))$. When $g$ is univalent, this condition is equivalent to $(0) = g(0)$ and
$f(U) \subseteq g(U)$.

The Bernoulli lemniscate is an important geometric domain in the study of analytic functions. It is associated with the mapping

$$(z) = \sqrt{1+z}, \qquad z\epsilon U. \qquad (1.3)$$

where the principal branch of the square root is considered. The image of the unit disk under this mapping is related to the right-hand part of the Bernoulli lemniscate, described by
$|w^2 - 1| < 1, \qquad R(w) > 0.$

The class of functions starlike with respect to symmetric points extends the classical notion of starlike functions by incorporating the symmetry of the values $(z)$ and $f(-z)$. This class has attracted considerable attention because it combines geometric symmetry with coefficient-theoretic properties. A normalized analytic function belongs to this class whenever

$$\frac{zf'(z)}{(z)-f(-z)} \prec \varphi(z) \qquad (1.4)$$

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where $\varphi$ is an analytic function with positive real part that maps the unit disk onto a prescribed geometric region. Functions associated with this domain have been investigated through subordination theory and coefficient estimates.

Recently, numerous subclasses have been investigated by choosing different comparison functions corresponding to various geometric domains, including the exponential, cardioid, lemniscate, and lune domains. Raina and Sokol in 2015 investigated coefficient estimates for a class of starlike functions associated with the lune-shaped mapping [1]. Their work provides a foundation for studying coefficient problems involving this geometric function. These domains produce distinct geometric behaviors and consequently lead to different coefficient estimates. Among them, the lune-shaped domain has emerged as an interesting object of study because of its simple analytic representation and rich geometric structure. Kumar et al. in 2013 [2] studied sufficient conditions for starlike functions associated with the Bernoulli lemniscate. More recently, in 2024 Nawaz et al. [3] investigated sharp coefficient estimates and Hankel determinants for classes of analytic functions related to this domain. Motivated by these developments, we consider the class of functions starlike with respect to symmetric points associated with the lemniscate of Bernoulli, defined by the subordination condition

$$\frac{2zf'(z)}{(z) - f(-z)} \prec \sqrt{1 + z}, \quad z \in U. \qquad (1.5)$$

Equivalently, there exists a Schwarz function $w$ such that

$$f(z\, zf'(z)) - f(-z) = w(z) + \sqrt{1 + w(z)^2} \qquad (1.6)$$

This representation establishes a direct connection between the coefficients of the function $f$ and the coefficients of the Schwarz function $w$. By expanding both sides into power series and comparing corresponding coefficients, one can express the Taylor coefficients $a_{2,}, a_{3,}a_{4,} \ldots$ in terms of the Schwarz coefficients $c_{2,}, c_{3,}c_{4,} \ldots$.
Subsequently, classical inequalities for Schwarz functions, including the Schwarz–Pick inequality and related coefficient estimates, can be employed to derive coefficient bounds for the proposed class.
Another important aspect of coefficient theory concerns logarithmic coefficients. For a normalized univalent function $f$ the logarithmic coefficients $\Upsilon_k$ are defined by

$$\frac{1}{2} \log\left(\frac{(z)}{z^K}\right) = \sum_{k=1}^{K} \Upsilon \frac{}{z} \qquad (1.7)$$

These coefficients have received considerable attention because they are closely connected with several extremal problems in geometric function theory and often

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provide finer information than the ordinary Taylor coefficients. Zaprawa (2021) obtained bounds for initial logarithmic coefficients of functions starlike and convex with respect to symmetric points [4]. This provides relevant background for investigating logarithmic coefficients in the proposed lune-associated class [5]. The present investigate the initial Taylor and logarithmic coefficient problems for functions starlike with respect to symmetric points associated with the lemniscate of Bernoulli.
Following lemma is use to prove our result

**Lemma 1:** [6] Let $(z) = c_1 z + c_2 z^2 + \cdots$ be a Schwarz function. Then, for any real numbers
μ and $v$ such that

$(\mu, v) \in \{|\mu| \leq \frac{1}{2}, -1 \leq v \leq 1\} \cup \{\frac{1}{2} \leq |\mu| \leq 2, \frac{4}{27}(|\mu| + 1)^3 - (|\mu| + 1) \leq v \leq 1\},$
the following sharp estimate holds

$|c_3 + \mu c_1 c_2 + v c^3| \leq 1.$
From the Schwarz-Pick Lemma, it follows that $\omega \in \mathscr{B}_0^1$ of the form (1.2)

$|c_2| \leq 1 - |c_1|^2.$

This inequality can be improved [7] as follows. For any $\lambda \in \mathbb{C}$,

$|c_2 + \lambda c^2| \leq ma\{1, |\lambda|\}.$ 1

Carlson in [8] obtained another generalization of the Schwarz-Pick Lemma.

1. Main Results:

Consider the class
$S^*(\varphi_L zf'(z)$
$) = f \epsilon A: \frac{f(z) - f(-z)}{}$
$\prec \varphi_L(z)$ s

We start with the coefficients of $f \epsilon S_s^*(\varphi_L$
$)$. Applying in

$f(z$
$zf'(z)$
$) - f(-z)$

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$= (z) + \sqrt{1 + w(z)^2}$ .

The expansions of $f$ and $w$ given by (1.1) and (1.2), we obtain

**Theorem1:** If $f \in S^*_{s,L}(\varphi_L)$ and is of the form (1.1), where

$$\frac{2zf'(z)}{(z) - f(-z)} \prec \sqrt{1 + z}$$

then the following sharp coefficient estimates hold:

$$|a_2| \leq \frac{1}{4}, \quad |a_3| \leq \frac{1}{4}, \quad |a_4| \leq \frac{1}{8}$$

**Proof**: By the definition of subordination, there exists a Schwarz function defined in (1.2) and

$$\frac{2zf'(z)}{(z) - f(-z)} = \sqrt{1 + w(z)} \qquad (2.1)$$

Now expanding both sides of (2.1) and then comparing coefficients we have

$$a_2 = \frac{1}{4} c_1,$$

$$a_{-} = -c_{-} \frac{1}{c^2}$$

(2.2)

$$a_4 = \frac{1}{8} c_3 - \frac{1}{32} c_1 c_2 + \frac{1}{128} c_1^3$$

Now taking modulus of expression (2.2) and applying Schwarz lemma we have

$$|a_2| \leq \frac{1}{4}$$

This result is sharp for the function $(z) = z$. Here $c_1 = 1$, $_2 = c_3 = 0$

From lemma 1, we have

Taking $\lambda = -1$

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we have $|a_3$

1

$| \leq \frac{1}{4}$.

$|c_2 + \lambda c^2| \leq ma\{1, |\lambda|\}$.

This result is sharp for the function $(z) = z^2$. Here $c_2 = 1$, $_1 = c_3 = 0$

Now we apply the Prokhorov–Szynal inequality from lemma 1, take

$\mu = \frac{-1}{4}$. $v = \frac{1}{16}$,

we have the required result. This result is sharp for the function $(z) = z^3$.

**Theorem2**: **:** If $f \in S_L^*$ $(\varphi_L)$ and is of the form (1.1), where

$$\frac{2zf'(z)}{(z) - f(-z)} \prec \sqrt{1 + z}.$$

If the logarithmic coefficients are defined by (1.7) then

$|\Upsilon_1| \leq \frac{1}{2}$, $|\Upsilon_2| \leq \frac{1}{8}$, $|\Upsilon_3| \leq \frac{1}{16}$

**Proof:** Expanding the logarithmic coefficients defined in (1.7), we have

$\Upsilon_1 = \frac{a_2}{2}$

$1 \; a^2 \Upsilon_2$

$= -(a_3 \; 2$

$1$

$- \frac{2}{2})$ (2.3)

$a^3$

$\Upsilon_3$

$= -(a_4 \; 2$

$- a_2$

$a_3$

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$+\frac{2}{})$ 3Substituting (2.2) in (2.3) we have

$\gamma_1 = \frac{1}{8} c_1$ –

$\gamma = -c$

$-\frac{3}{} c^2$

(2.4)

$\gamma = \frac{1}{} c$

$\frac{}{64}$

$-\frac{3}{} c\ c$

$+\frac{11}{3} c^3$ — — —

16 3 64 1 2 768 1

Taking absolute values and applying the Schwarz lemma we have $|\gamma_1| \leq \frac{1}{8}$ Now by applying –.

lemma 1 (Cauchy Schwarz inequality) taking $\lambda = \frac{-3}{8}$, we have $|\gamma_2| \leq -$. Next for bound of ,we 1 3

apply the Prokhorov–Szynal inequality for Schwarz functions from lemma 1 which is

$|c_3 + \mu c_1 c_2 + \nu c^3| \leq 1.$ 1

Here $\mu = \frac{-3}{4}$. $\nu = \frac{11}{48}$, we have the required result. These three bounds are sharp, with equality attained by the corresponding Schwarz functions $w(z) = z$, $w(z) = z^2$, $w(z) = z^3$, respectively.

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