Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 3 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189

http://amresearchreview.com/index.php/Journal/about

Hankel and Inverse-Coefficient Bounds for Functions with Bounded Turning

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ABSTRACT

In this paper, we investigate coefficient problems for the class of analytic functions with bounded turning, defined by the condition $Re\, f'(z) > 0$ in the open unit disk. The study is based on the representation of the derivative of a normalized analytic function through the Carathéodory class of functions with positive real part. By employing classical coefficient inequalities and suitable coefficient-functional estimates, we derive bounds for the initial Taylor coefficients and investigate the corresponding inverse coefficients. Furthermore, we examine Hankel determinants.

Keywords: Analytic Functions, Bounded Turning, Carathéodory Functions, Taylor Coefficients, Inverse Coefficients, Hankel Determinants

Introduction:

The theory of analytic and univalent functions is an important area of complex analysis and geometric function theory. A considerable part of this theory is concerned with investigating the Taylor coefficients of normalized analytic functions and establishing bounds for different coefficient functional. These problems have attracted considerable attention because coefficient estimates provide useful information about the geometric behavior of analytic mappings, including their growth, distortion, and univalence properties.

Let $U = \{z: |z| < 1\}$ denote the open unit disk, and let A represent the family of normalized analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_{\kappa}\, z^{\kappa} \qquad (1.1)$$

satisfying the normalization conditions $f(0) = f'(0) - 1 = 0$. The coefficients of such functions encode important geometric information about their image domains and mapping properties. Consequently, coefficient estimates and coefficient functional have remained fundamental themes in the theory of univalent functions [1, 2].

One of the classical subclasses of normalized analytic functions is the class of

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functions with bounded turning, commonly denoted by R. It is defined by

$$\mathcal{R} = \{f \in \mathcal{A}: Re\, f'(z) > 0, z \in \mathbb{D}\}. \qquad (1.2)$$

The positivity of the real part of the derivative implies that f is close-to-convex and, in particular, univalent in U, by the classical univalence criterion of Noshiro and Warschawski; see, for example, Duren [1]. This condition also provides a direct analytic link between the function f and the well-studied family of analytic functions having positive real part. The Carathéodory class $\mathcal{P}$ consists of functions analytic in U satisfying $Re\, p(z) > 0$ and $p(0) = 1$. Each $p \in \mathcal{P}$ has an expansion

$$p(z) = 1 + c^1 z + c^2 z^2 + \cdots. \qquad (1.3)$$

For $f \in \mathcal{R}$, the defining condition allows us to write $f'(z) = p(z)$ for some $p \in \mathcal{P}$. Gregory coefficients are decreasing rational numbers 1/2, 1/12, 1/24, 19/720, … . which are similar in their role to the Bernoulli numbers and appear in a large number of problems, especially in those related to the numerical analysis and to the number theory. They first appeared in the works of the Scottish mathematician James Gregory in 1671 [3] and were subsequently rediscovered many times. Among the famous mathematicians who rediscovered them, we find Laplace, Mascheroni, Fontana, Bessel, Clausen, Hermit, Pearson and Fisher. Furthermore, Gregory's coefficients can rightly be considered one of the most frequently rediscovered in mathematics (the last rediscovery dates back to our century), the reason for which in literature we can find them under various names (e.g. reciprocal logarithmic numbers, Bernoulli numbers of the second kind, Cauchy numbers, etc.). For the same reason, their authorship is often attributed to various mathematicians.

The inverse-function problem is a natural continuation of the coefficient. Since every function in $\boldsymbol{\mathcal{R}}$ is univalent, its local inverse exists near the origin and has an expansion of the form

$$f^{-1}(w) = w + A^2 w^2 + A^3 w^3 + A^4 w^4 + \cdots. \qquad (1.4)$$

Löwner [4] proved that, $if\ f \in S$ and its inverse is given by (1.4), then the sharp estimate

$$|A_n| \leq \frac{(2n)!}{n!\,(n+1)!} \qquad (1.5)$$

holds. It has been shown that the inverse of the Koebe function $k(z) = \frac{z}{(1-z)^2}$ provides the best bounds for all $|A_n|$ $(n = 2, 3, \dots)$ in (1.5) over all members of S. There has been a good deal of interest in determining the behavior of the inverse coefficients of f given in (1.5) when the corresponding function f is restricted to some proper geometric subclasses of S. Alternate proofs of the inequality (1.5) have been given by several authors but a simpler proof was given by Yang [5].

The class and its subordination representation:

Let f belongs to Class R as defined in (1.2) and of the form (1.1) and suppose $f'(z) \prec \frac{1+z}{1-z}$,

For analytic functions f and g in U, we say that f is subordinate to g, written as $f \prec g$, if there exists a Schwarz function

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$$w(z) = \sum_{k=1}^{\infty} c_{\text{K}}\, z^{\text{K}} \quad , |w(z)| < 1, \quad w(0) = 0. \qquad (2.1)$$

such that $f(z) = g(w(z))$. When g is univalent, this condition is equivalent to $f(0) = g(0)$ and $f(U) \subseteq g(U)$. So by subordination function there exist a Schwarz function such that

$$f'(z) = \frac{1 + w(z)}{1 - w(z)} \qquad (2.2)$$

As $f'(z) = p(z)$. Define

$$p(z) = \frac{1 + w(z)}{1 - w(z)} \qquad (2.3)$$

Now comparing coefficients, we have

$$a_2 = \frac{p_1}{2}$$
$$a_3 = \frac{p_2}{3} \qquad (2.4)$$
$$a_4 = \frac{p_3}{4}$$
$$a_5 = \frac{p_4}{5}$$
$$a_6 = \frac{p_5}{6}$$

The following lemmas is needed for the main results.

Lemma 2.1: [6] *If*

$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}\ (p_1 \geq 0)$,

then

$$2p_2 = {p_1}^2 + x(4 - {p_1}^2)$$
$$4p_3 = {p_1}^3 + 2(4 - {p_1}^2)p_1 x - p_1(4 - {p_1}^2)x^2 + 2(4 - {p_1}^2)(1 - |x|^2)y$$

for some $x, y \in \mathbb{C}$ *with* $|x| \leq 1$ *and* $|y| \leq 1$.

Lemma 2.2 [7] *If* $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}\ (p_1 \geq 0)$, *then*

$$|p_{\text{n}}| \leq 2 \quad (n \geq 1)$$

and if $Q \in [0, 1]$ *and* $Q(2Q - 1) \leq R \leq Q$, *then*

$|p_3 - 2Qp_1p_2 + Rp_1{}^3| \leq 2$.

Also

$|p_{n+k}$ –

$\mu p_{\text{n}}p_{\text{k}}| \leq 2\ max\{1, |2\mu - 1|\}$

$$= 2\begin{cases} 1, & for\ 0 \leq \mu \leq 1 \\ |2\mu - 1|, & otherwise \end{cases}$$

The inequalities are taken from [8] [6] and [8] respectively.

Main Results:

Theorem3.1: Let f belongs to Class R and of the form (1.1) and suppose $f'(z) \prec \frac{1+z}{1-z}$,

then

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$$|a_2| \leq 1, |a_3| \leq \frac{2}{3}, |a_4| \leq \frac{1}{4}, |a_5| \leq \frac{2}{5}, |a_6| \leq \frac{1}{3}.$$

All five estimates are sharp.

Proof: Taking modulus of (2.1) and applying Lemma 2 we have the reqired result Therefore

$$|a_n| \leq \frac{2}{n}, n = 2,3,4,5,6$$

These estimates are sharp for the function $f(z) = z + \sum_{n=2}^{\infty} \frac{2}{n} z^n$.

Theorem3.2: Let f belongs to Class R and of the form (1.1) and suppose $f'(z) \prec \frac{1+z}{1-z}$,

Then the second Hankel determinants satisfies

$$|H_2(2)| = |a_2 a_4 - a_3^2| \leq \frac{4}{9}.$$

The bound is sharp.

Proof: As

$$|H_2(2)| = |a_2 a_4 - a_3^2|$$

From (2.1) we have

$$|H_2(2)| = \left| \frac{p_1 p_3}{5} - \frac{{p_2}^2}{9} \right|$$

By applying lemma 1, we have

$$|H_2(2)| = \left| \frac{c^4}{288} + \frac{c^2(4-c^2)x}{144} - \frac{(c^2+32)(4-c^2)x}{288} + \frac{c(4-c^2)(1-|x^2|)y}{16} \right|.$$

Let $\rho = |x|, 0 \leq \rho \leq 1, \ |y| \leq 1$. And applying triangular inequality we have

$$|H_2(2)| \leq F(\rho).$$

Where

$$F(\rho) = \frac{c^4}{288} + \frac{c(4-c^2)}{16} + \frac{c^2(4-c^2)\rho}{144} + \frac{(c^2+32-18c)(4-c^2)\rho^2}{288}.$$

Clearly $F'(\rho)$ is an increasing function and its maximum occur at $\rho = 1$.

Consequently

$$|H_2(2)| \leq F(1).$$

After some simplification we have the required result. And this bound is sharp for the function $\frac{1+z^2}{1-z^2}$.

Theorem 3.3: Let f belongs to Class R and of the form (1.1) and suppose $f'(z) \prec \frac{1+z}{1-z}$,

Suppose the inverse function has the expansion

$$f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + A_4 w^4 + \cdots.$$

Then the following estimates hold

$$|A_2| \leq 1, |A_3| \leq \frac{4}{3}, |A_4| \leq \frac{13}{6}.$$

Proof: Since $f \ \epsilon R$ we have

$$Ref'(z) > 0$$

Using the identity $f\big(F(w)\big) = w$, and by (1.5)

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$$\begin{aligned} A_2 &= -a_2 \\ A_3 &= 2a_2^3 - a_3 \\ A_4 &= -5a_2^3 + 5a_2a_3 - a_4. \end{aligned} \tag{3.1}$$

Substituting (2.1) in (3.1) we have

$$\begin{aligned} A_2 &= -\frac{p_1}{2} \\ A_3 &= \frac{p_1^2}{2} - \frac{p_2}{3} \\ A_4 &= -\frac{5p_1^3}{8} + \frac{5p_1p_2}{6} - \frac{p_3}{4} \end{aligned} \tag{3.2}$$

Now applying lemma 2 we have the required results.

References:

1. Duren, P.L., *Univalent functions*. Vol. 259. 2001: Springer Science & Business Media.
2. Pommerenke, C., *On the Hankel determinants of univalent functions.* Mathematika, 1967. **14**(1): p. 108-112.
3. Phillips, G., *Gregory's method for numerical integration.* The American Mathematical Monthly, 1972. **79**(3): p. 270-274.
4. Löwner, K., *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I.* Mathematische Annalen, 1923. **89**(1): p. 103-121.
5. Yang, S., *Estimates for coefficients of univalent functions from integral means and Grunsky inequalities.* Israel Journal of Mathematics, 1994. **87**(1): p. 129-142.
6. Libera, R.J. and E.J. Złotkiewicz, *Coefficient bounds for the inverse of a function with derivative in* Proceedings of the American Mathematical Society, 1983. **87**(2): p. 251-257.
7. Kazımoğlu, S., E. Deniz, and H.M. Srivastava, *Sharp Coefficients Bounds for Starlike Functions Associated with Gregory Coefficients: S. Kazımoğlu et al.* Complex Analysis and Operator Theory, 2024. **18**(1): p. 6.
8. Pommerenke, C., *Univalent functions.* Vandenhoeck and Ruprecht, 1975.