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# On Taylor and Logarithmic Coefficient Bounds for Functions Starlike with Respect to Symmetric Points Associated with the Lune

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## ABSTRACT

This paper investigates Taylor and logarithmic coefficient bounds for a class of analytic functions that are starlike with respect to symmetric points and associated with the lune-shaped domain. The class is defined through a subordination condition involving the conformal function $\varphi_L(z) = z + \sqrt{1+z^2}$, which provides a geometric framework for studying the behavior of normalized analytic functions in the open unit disk. By employing the subordination principle and the coefficient expansion of an associated Schwarz function, we derive relations between the Taylor coefficients of the functions under consideration and the coefficients of the Schwarz function. These relations are then used to establish upper bounds for the initial Taylor coefficients. Furthermore, by expanding the logarithm of the normalized function, we obtain expressions for the initial logarithmic coefficients and investigate their corresponding bounds. The results contribute to the study of coefficient problems in geometric function theory and provide a framework for examining how the geometry of the lune-shaped domain influences Taylor and logarithmic coefficients. The estimates are presented with detailed derivations, while questions of sharpness are considered separately.

**Keywords:** Geometric function theory; analytic functions; starlike functions with respect to symmetric points; lune-shaped domain; subordination; Taylor coefficients; logarithmic coefficients; Schwarz functions.

## Introduction:

Geometric Function Theory (GFT) is one of the most active branches of complex analysis, dealing with the geometric properties of analytic and univalent functions defined in the open unit disk. A fundamental area of research in GFT concerns the investigation of subclasses of normalized analytic functions characterized by

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geometric conditions such as starlikeness, convexity, close-to-convexity, and subordination. In particular, determining sharp coefficient estimates for these subclasses has remained an important research problem because the Taylor coefficients provide significant information about the geometric behavior of analytic functions.

Let $U = \{z: |z| < 1\}$ denote the open unit disk, and let $A$ represent the family of normalized analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_{\mathrm{K}}\, z^{\mathrm{K}} \qquad (1.1)$$

satisfying the normalization conditions $f(0) = f'(0) - 1 = 0$.

The coefficient problem consists of determining the best possible bounds for the Taylor coefficients $a_{\mathrm{K}}$ and related coefficient functionals associated with various subclasses of $A$.Subordination theory provides one of the most effective techniques for studying coefficient problems. For analytic functions $f$ and $g$ in $U$, we say that $f$ is subordinate to $g$, written as $f \prec g$, if there exists a Schwarz function

$$w(z) = \sum_{k=1}^{\infty} c_{\mathrm{K}}\, z^{\mathrm{K}} \quad , |w(z)| < 1, \quad w(0) = 0. \qquad (1.2)$$

such that $f(z) = g(w(z))$. When $g$ is univalent, this condition is equivalent to $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

The class of functions starlike with respect to symmetric points extends the classical notion of starlike functions by incorporating the symmetry of the values $f(z)$ and $f(-z)$. This class has attracted considerable attention because it combines geometric symmetry with coefficient-theoretic properties. A normalized analytic function belongs to this class whenever

$$\frac{zf'(z)}{f(z) - f(-z)} \prec \varphi(z) \qquad (1.3)$$

where $\varphi$ is an analytic function with positive real part that maps the unit disk onto a prescribed geometric region.

Recently, numerous subclasses have been investigated by choosing different comparison functions corresponding to various geometric domains, including the exponential, cardioid, lemniscate, and lune domains. Raina and Sokol (2015)

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investigated coefficient estimates for a class of starlike functions associated with the lune-shaped mapping [1]. Their work provides a foundation for studying coefficient problems involving this geometric function. These domains produce distinct geometric behaviors and consequently lead to different coefficient estimates. Among them, the lune-shaped domain has emerged as an interesting object of study because of its simple analytic representation and rich geometric structure.

The Lune function is as:

$$\varphi_L(z) = z + \sqrt{1+z^2}\,, \qquad (1.4)$$

where the principal branch of the square root is chosen so that $\sqrt{1} = 1$.

The function $\varphi_L$ maps the unit disk onto a lune-shaped region that is symmetric with respect to the real axis. This mapping has been successfully employed in the study of several subclasses of starlike and convex functions, particularly in obtaining coefficient estimates through subordination techniques.

Motivated by this geometric mapping, we introduce the class of functions starlike with respect to symmetric points associated with the lune, denoted by $S^*_{ss}(\varphi_L)$, consisting of all functions $f \in A$ satisfying

$$\frac{zf'(z)}{f(z)-f(-z)} \prec z + \sqrt{1+z^2}, \quad z\epsilon U \qquad (1.5).$$

Equivalently, there exists a Schwarz function $w$ such that

$$\frac{zf'(z)}{f(z)-f(-z)} = w(z) + \sqrt{1+w(z)^2} \qquad (1.6)$$

This representation establishes a direct connection between the coefficients of the function $f$ and the coefficients of the Schwarz function $w$. By expanding both sides into power series and comparing corresponding coefficients, one can express the Taylor coefficients $a_{2,}, a_{3,} a_{4,}$ ... in terms of the Schwarz coefficients $c_{2,}, c_{3,} c_{4,}$ ....
Subsequently, classical inequalities for Schwarz functions, including the Schwarz–Pick inequality and related coefficient estimates, can be employed to derive coefficient bounds for the proposed class.
Another important aspect of coefficient theory concerns logarithmic coefficients. For a normalized univalent function $f$ the logarithmic coefficients $\Upsilon_k$ are defined by

$$\frac{1}{2} log\left(\frac{f(z)}{z}\right) = \sum_{k=1}^{\infty} \Upsilon_{\mathrm{K}}\, z^{\mathrm{K}} \qquad (1.7)$$

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These coefficients have received considerable attention because they are closely connected with several extremal problems in geometric function theory and often provide finer information than the ordinary Taylor coefficients. Zaprawa (2021) obtained bounds for initial logarithmic coefficients of functions starlike and convex with respect to symmetric points [2]. This provides relevant background for investigating logarithmic coefficients in the proposed lune-associated class

The present work is motivated by the coefficient-estimation framework developed for functions starlike with respect to symmetric points [3] and extends this approach to the lune-associated class. The principal objective of this paper is to establish coefficient bounds for functions belonging to $S_{ss}^{*}(\varphi_L)$. Specifically, we first derive explicit relationships between the Taylor coefficients and the Schwarz coefficients through coefficient comparison. These relationships are then used to obtain bounds for the initial Taylor coefficients. Furthermore, we investigate the corresponding logarithmic coefficients and establish their estimates using suitable inequalities for Schwarz functions.

**Lemma 1:** [4] If $p \in P$, then $|p_n| \leq 2, \forall n \in N$.

**Lemma 2: [5]** Let $\omega(z) = c_1 z + c_2 z^2 + \cdots$ be a Schwarz function. Then, for any real numbers μ and $v$ such that

$(\mu, v) \in \left\{|\mu| \leq \frac{1}{2}, -1 \leq v \leq 1\right\} \cup \left\{\frac{1}{2} \leq |\mu| \leq 2, \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \leq v \leq 1\right\}$,

the following sharp estimate holds

$$|c_3 + \mu c_1 c_2 + v c_1^3| \leq 1.$$

From the Schwarz-Pick Lemma, it follows that $\omega \in \mathcal{B}_0$ of the form (1.2)

$$|c_2| \leq 1 - |c_1|^2.$$

This inequality can be improved (see [6] ) as follows. For any $\lambda \in \mathbb{C}$,

$$|c_2 + \lambda c_1^2| \leq max\{1, |\lambda|\}.$$

Carlson in [7] obtained another generalization of the Schwarz-Pick Lemma.

## 1. Coefficient Bounds:

Consider the class

$$S_s^{*}(\varphi_L) = f \epsilon A: \frac{z f'(z)}{f(z) - f(-z)} \prec \varphi_L(z)$$

We start with the coefficients of $f \epsilon S_s^{*}(\varphi_L)$. Applying in

$$\frac{z f'(z)}{f(z) - f(-z)} = w(z) + \sqrt{1 + w(z)^2}.$$

The expansions of $f$ and $w$ given by (1.1) and (1.2), we obtain

$$a_2 = \frac{c_1}{2}, \qquad (2.1)$$

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$$a_3 = \frac{c_2}{2} + \frac{c_1^2}{4} \qquad (2.2),$$

$$a_4 = \frac{c_3}{4} + \frac{3c_1c_2}{8} + \frac{c_1^3}{16} \qquad (2.3).$$

**Theorem1:** If $f \epsilon S_S^*(\varphi_L)$ and is of the form (1.1), then the following coefficient estimates hold:

$$|a_2| \leq \frac{1}{2},$$

$$|a_3| \leq \frac{1}{2},$$

$|a_4| \leq \frac{1}{4}$.

Proof: Taking modulus of expression (2.1) and applying lemma 1 we obtain $|a_2| \leq \frac{1}{2}$.

Now taking modulus of expression (2.2) and applying triangular inequality we have

$|a_3| \leq \frac{|c_2|}{2} + \frac{|c_1^2|}{4}$.

The Schwarz–Pick coefficient inequality gives

$|a_3| \leq \frac{1}{2} - \frac{|c_1^2|}{4} \leq \frac{1}{2}$.

Now taking modulus of expression (2.3) we have

$$|a_4| = \frac{1}{4}\left|c_3 + \frac{3c_1c_2}{2} + \frac{c_1^3}{4}\right|$$

Applying lemma 2 we have the required result.

The logarithmic coefficients are defined in (2.7). If $f$ is given by (1.1), then its logarithmic coefficients are given as follows:

$$\Upsilon_1 = \frac{a_2}{2}, \qquad (2.4)$$

$$\Upsilon_2 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{2}\right) \qquad (2.5),$$

$$\Upsilon_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{a_2^3}{3}\right) \qquad (2.6)$$

**Theorem2**: **:** If $f \epsilon S_S^*(\varphi_L)$ and is of the form (1.1), then

$|\Upsilon_1| \leq \frac{1}{4}, \quad |\Upsilon_2| \leq \frac{1}{4}, \quad |\Upsilon_3| \leq \frac{13}{96} + \frac{1}{24\sqrt{3}}$

Proof: Applying (2.1), (2.2) and (2.3) in (2.4), (2.5) and (2.6), we have

$$\Upsilon_1 = \frac{c_1}{4}$$

$$\Upsilon_2 = \frac{c_2}{4} + \frac{c_1^2}{16}$$

$$\Upsilon_3 = \frac{c_3}{8} + \frac{c_1c_2}{16} - \frac{c_1^3}{96}.$$

Taking absolute values and applying lemma 2 we have the required results.

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