Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 3 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189
http://amresearchreview.com/index.php/Journal/about

Analysis of Backward Source Problems for Some Fractional Order Differential Equations

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ABSTRACT

In this article, we intend to look at certain backward source problems for Fractional Differential Equations (FDEs) with Robin boundary conditions, by considering a bi-ordinal Hilfer fractional derivative ${}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}$, *of order* $\vartheta: (n-1<\vartheta\leq n, \quad n\in N)$ *and* $\chi: (n-1<\chi\leq n, \quad n\in N)$ *of type* $\zeta: (0\leq\zeta\leq 1)$, *(for both space and time). We show that the backward problem is well-posed by expanding the eigenfunctions of a spectral problem and using the generalized Fourier method.*

Keywords: *Backward source problem Fractional operator Eigenvalues Bi-orthogonal system Mittag-Leffler functions and generalization [2022] 26A33 35R30 35P15 34M50 42A16*

Introduction

The fractional integral and derivative can be considered as an interpolation of an infinite sequence of classical n-fold integrals and n-fold derivatives. Fractional calculus is the area of calculus that has been rapidly developing over the last couple of centuries. It offers numerous new features for research and thus becoming increasingly popular in a variety of applications, Abel uses the application of semi-derivative for the first time, which is related to the study of curves, then other researchers utilized it in different domains including geophysics, astronomy, weather forecasting, oceanography, hydrology, petroleum engineering, electromagnetic, fluid mechanics, biological population models, optics and signals processing discussed in [1, 2, 3, 4, 5,

⋆Analysis of Backward Source Problems for Some Fractional Order Differential Equations.

6, 7, 8, 9] respectively. It has been documented by a number of researchers that the nonstandard behavior of diffusion and transport phenomena has been observed in both the theoretical fields and laboratory tests, for instance, [10, 11]. The non-locality of fractional operators is the driving force behind the modeling success of fractional differential equations (FDEs). Due to this property, these operators can adequately describe the non-local effects and long memories that are typical of the majority of physical phenomena [12]. Backward problems are among the preponderance important in mathematical problems and mathematics. The backward problem gained less attention as compared to the direct problem. For our purpose,

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we concentrate on the literature regarding the backward source problem for FODE's. There are many classifications of backward problems, for instance, backward source problem, evolutionary backward problem, backward coefficient problem, boundary value problem, and many more.

There have been few research looking at the backward problems of obtaining the coefficients in the heat equation for instances, in which the unknown coefficient is space-dependent in [13] both temporal and space-dependent [14, 15]. The recovery of the initial condition for a Time Fractional Diffusion Equation (TFDE) using additional over-determination data is assessed in [16] article. Initial data was recreated by utilizing over-determination data acquired from the geographic domain explored in [17]. A regularization technique for the Inverse Source Problem (ISP) of a Space-Fractional Diffusion Equation (SFDE) was proposed in [18], by utilizing a space-fractional derivative defined in the Riesz-Feller space-fractional derivative. The inverse problems of the viscoelastoplastic and torsional equations of temporal fractional viscoelasticity have been achieved numerically in [19] and [20], respectively. For an ISP, by using Carleman estimations [21], to produce Lipschitz-type stability estimates.

For SFDE, the inversion of diffusion coefficients has only been discussed over just a pieces of literature. Using the eigenfunction expansion approach, various unique conclusions (with distinct situations) for an ISP of time fractional mixed parabolic-hyperbolic type equation is demonstrated in [22]. The unique result of the time-dependent source term in a Space-Time Fractional Diffusion Equation (STFDE) is determined in [23]. Inverse Coefficient Problem (ICP) was evaluated for a nonlinear TFDE in [24, 25]. With the help of a non-self-adjoint operator associated with the FDE method, eigenfunctions, and eigenvalues of the Boundary Value Problem (BVP) for fractional oscillator equation, mentioned in [26].

In particular, we analyze two backward source problems of a fractional diffusion equation under the premise that the unknown source term is independent of time and space, respectively. A logical result may be obtained by using the eigenfunction expansion principle.

The remaining portions of the article are structured as follows: In section 2, a statement of the problem is discussed as well as over-determination conditions for both problems are also analyzed. Section 3 is devoted to Prolegomena, in which some basic concepts of fractional calculus and some results related to the Mittag-Leffler function are mentioned there. The spectral problem and its conjugate problem of the system are described in Section 4. In Section 5, we establish the main results concerning the existence and uniqueness of the solution of a backward source problem.

Statement of the problem

For the following TFDE, a backward source problem was explored.

$$ {}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta} z(y,t) = z_{yy}(y,t) + F(y,t), \quad (y,t) \in \Psi, \qquad (1)$$

along with initial data

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$$I_{0+,t}^{(1-\zeta)(1-\chi)}T_n(t)|_{t=0} = \phi_n(y), \qquad (2)$$

and boundary data

$$z(0,t) = 0, \quad u_y(0,t) + z_y(\pi,t) + \rho u(\pi,t) = 0, \quad \rho > 0, \qquad (3)$$

where, $\Psi := (y,t): 0 < y < 1, t \in (0,T]$, *and* ${}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}$ *represents left sided bi-ordinal HFD of order* $\vartheta: (n-1 < \vartheta \le n, n \in N)$ *and* $\chi: (n-1 < \chi \le n, n \in N)$ *of type* $\zeta: (0 \le \zeta \le 1)$.

In the first problem, $F(y,t) := p(t)f(y)$ *here,* $F(y,t)$ *is the source term in which* $p(t)$ *is an unknown function and* $f(y)$ *is known function. We recover a set of function* $\{z(y,t), p(t)\}$ *for the regular solution, only if the total energy of the system is known, i.e.,*

$$\int_0^\pi z(y,t)dy = E(t). \qquad (4)$$

In the second problem, $F(y,t) := f(y)g(y,t)$ *here,* $g(y,t)$ *is known function and* $f(y)$ *is to be investigated. To determine a regular solution, we intend to recover a couple of function* $\{z(y,t), f(y)\}$ *in the presence of an additional condition*

$$z(y,t) = \psi(y). \qquad (5)$$

Prolegomena

Some fundamental concepts about fractional calculus and few of its properties are discussed in this Section.

Definition 3.1 *[12] Let* $g \in L^1_{loc}[a,b] where -\infty < a < u < b < \infty$, *then RLFI of order* ϑ *are defined as*

$$I_{a+,u}^{\vartheta}g(u) := \frac{1}{\Gamma(\vartheta)}\int_a^u \frac{g(\tau)}{(u-\tau)^{1-\vartheta}d\tau}, \quad u > a,$$

$$I_{b-,u}^{\vartheta}g(u) := \frac{1}{\Gamma(\vartheta)}\int_a^u \frac{g(\tau)}{(\tau-u)^{1-\vartheta}d\tau}, \quad u < b,$$

For $\vartheta = n \in N$, *RLFI(left sided) interpolates integrals in classical integer order except that the domain changes from Riemann integrable to Lebesgue integral functions.*

Definition 3.2 *[27] Left-sided bi-ordinal Hilfer fractional derivative defined as:*

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$$ {}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}g(u) = I_{0+,t}^{\zeta(n-\vartheta)}\frac{d^n}{dt^n}g(u)I_{0+,t}^{(1-\zeta)(n-\chi)}g(u), $$

of order ϑ: $(n-1<\vartheta\le n, n\in N)$ *and* χ: $(n-1<\chi\le n, n\in N)$ *of type* $\zeta(0\le\zeta\le 1)$.

Remark 3.1 *Formula for the Laplace transform of the bi-ordinal Hilfer fractional derivative*

$$ \mathcal{L}\{{}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}g(u)\} = s^{\chi+\zeta(\vartheta-\chi)}\mathcal{L}\{g(u)\} - s^{1-\zeta(2-\vartheta)}[I_{0+}^{(1-\zeta)(2-\chi)} $$

$$ +g(u)|_{t\to 0+}] - s^{-\zeta(2-\vartheta)}\left[\frac{d}{dt}I_{0+}^{(1-\zeta)(2-\chi)}g(u)|_{t\to 0+}\right]. $$

Definition 3.3 *[28] The Mittag-Leffler function for* ϑ,χ *two parameters with* $Re(\vartheta)\ge 0$ *and* $\chi\in C$ *is defined as*

$$ E_{\vartheta,\chi}(u) = \sum_{k=0}^{\infty}\frac{u^k}{\Gamma(\vartheta k+\chi)}, \quad Re(\vartheta)>0, \quad u,\vartheta\in C. $$

Lemma 3.1 *[29] If* ϑ *is an arbitrary real number,* $\chi<2$*,* μ *is such that* $\frac{\pi\alpha}{2}<\mu<min\pi,\pi\chi$, $u\in C$ *such that* $|u|\ge 0$, $\mu\le|arg(u)|\le\pi$ *and* C_1 *is a real constant, then*

$$ |E_{\vartheta,\chi}(u;1)| \le \frac{C_1}{1+|u|}. $$

Lemma 3.2 *[30] For* $g(u)\in C[a,b]$ *and* $\mu\in R$*, the following relation holds*

$$ |g(u)*e_{\vartheta,\chi}(u;\pm\mu)| \le \frac{C_1}{\mu}|g(u)|, $$

where C_1 *is a constant as aforementioned.*

Lemma 3.3 *[30] The* $e_{\vartheta,\vartheta+1}(u;\pm\mu)$ *is Mittag-Leffler type functions which satisfy the relation*

$$ e_{\vartheta,\vartheta+1}(\pm\mu u^{\vartheta}) = \frac{1}{\mu}(\pm 1\mp e_{\vartheta,1}(u;\pm\mu)), \quad \mu>0. $$

Lemma 3.4 *[28] The Mittag-Leffler function* $e_{\vartheta,\chi(u;\mu)}$*, for* $\mu>0$, $0<\vartheta\le\chi\le 1$*, are entirely monotonous functions, i.e., we have*

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$$(-1)^k \frac{d^k}{du^k} e_{\vartheta,\chi}(u,\mu) \geq 0, \quad k \in N \cup 0.$$

Spectral problem

In this section, we will discuss spectral problem of (1)-(3). Furthermore, we present and prove two lemmata that will prove indispensable when demonstrating the convergence of the solutions.

To address backward problems, we will use separation of variables. For (1) and (3), the spectral problem is

$$I''(y) = -\lambda I(y), \qquad (6)$$

$$I(0) = 0, \quad I'(0) + I'(\pi) + \rho I(\pi) = 0. \qquad (7)$$

Eigenvalues of equation (6) to (7) are

$$\lambda_{1k} = (2k+1)^2, \quad \lambda_{2k} = (2\alpha_k)^2,$$

where, the non-linear equation $cot(\pi\zeta) = -\rho/(2\zeta)$ gives the solution α_k and fulfills $2k+1 < 2\alpha_k < 2k+2$. Eigenfunctions corresponding to eigenvalues λ_{1k} and λ_{2k} are

$$z_{1k}(y) = sin((2k+1)y), \quad z_{2k}(y) = sin(2\alpha_k y), \qquad (8)$$

respectively.

Due to the results of [26], eigenfunctions of (6) cannot be used to tackle the backward problems. So, introduced a conjugate problem correspondent to (6) and (7) is

$$J''(y) = -\lambda J(y),$$

$$J(0) + J(\pi) = 0, \quad J'(\pi) + \rho J(\pi) = 0.$$

Eigenvalues of the conjugate problem are identical to the (6) to (7). Eigenfunction correspondent to λ_{1k} and λ_{2k} are

$$v_{1k}(y) = sin((2k+1)y) - \frac{2k+1}{\rho} cos((2k+1)y),$$

and

$$v_{2k}(y) = C_{2k}\{sin(2\alpha_k y) - \frac{2\alpha_k}{\rho} cos(2\alpha_k y)\},$$

respectively,

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where C_{2k} is selected in such a manner that

$$\int_0^{\pi} z_{2k}(y) v_{2k}(y) dy = 1.$$

The following bi-orthogonal set of functions has been constructed with the assist of eigenfunctions of spectral and conjugate problems [26],

$$M_\rho = \{m_{2k}(y), m_{2k+1}(y)\}, \quad Q_\rho = \{q_{2k}(y), q_{2k+1}(y)\}, \qquad (9)$$

where

$$m_{2k}(y) = z_{1k}(y), \quad m_{2k+1}(y) = \frac{(z_{2k}(y) - z_{1k}(y))}{2\delta_k}, \qquad (10)$$

$$q_{2k}(y) = v_{2k}(y) + v_{1k}(y), \quad q_{2k+1}(y) = 2\delta_k v_{2k}(y),$$

and $\delta_k = \alpha_k - k - \frac{1}{2}$.

Riesz basis is used to get the set of functions M_ρ [26] and Q_ρ [31]. Where, notice that

$$\frac{1}{\lambda_{1k}} \le \frac{1}{k^2}, \quad \frac{1}{\lambda_{2k}} \le \frac{1}{k^2}, \ |\Delta_k| \le 7k. \qquad (11)$$

The following Lemma are indispensable while demonstrating the existence of a solution.

Lemma 4.1 *For $g \in C^2(0,\pi)$ such that $g^{(n)}(o) = 0 = g^{(n)}(\pi), n = 0,1$ then*

$$|g_k| \le \frac{1}{k^2} \langle g''(y), m_k(y)\rangle, \quad k \in \mathbb{N}.$$

Lemma 4.2 *For $g \in C^4(0,\pi)$ such that $g^{(n)}(0) = 0 = g^{(n)}(\pi), \quad n = 0,1,2$ then*

$$|g_k| \le \frac{1}{k^4} \langle g^{(4)}(y), M_k(y)\rangle, \quad k \in N.$$

Backward source problems

In this section, we will explore two backward source problems that have arisen in this article. Firstly, we will investigate the time-dependent backward source problem and secondly, space-dependent backward source problem is extracted, an integral

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Volume 4, Issue 3 (2026)

Online ISSN 3007-3197 Print ISSN 3007-3189
http://amresearchreview.com/index.php/Journal/about

type and final temperature over-determination condition is applied to demonstrated that the results of the backward problems exist, and unique.

Theorem 5.1 *Under the given assumptions, the backward source problem has a regular local solution.*

- $\phi \in C^2(0,\pi)$ *and* $\phi^{(n)}(0) = 0 = \phi^{(n)}(\phi), n = 0,1,2,3,$

- $f \in C^{2,0}_{y,t}(\psi)$ *and* $\frac{\partial^m f}{\partial y^m}(0,t) = 0 = \frac{\partial^m f}{\partial y^m}(\pi,t), m = 0,1,$ *moreover,*

$$0 \leq \frac{1}{M_1} \leq \int_0^\pi f(y)dy, \quad M_1 > 0,$$

- $E(t) \in AC(0,T)$ *along with satisfying the relation,* $\int_0^\pi \phi(y)dy = E(0)$.

Proof. We'll begin by constructing the series representation of $z(y,t)$ *and* $f(y)$*, then use the Banach Fixed Point Theorem to show that* $p(t)$ *exists uniquely. Moreover, existence and uniqueness of* $z(y,t)$ *are shown.*

Construction of the solution

Using the spectral problem (6), we will generate series representations of the unknown functions listed below

$$z(y,t) = \sum_{k=1}^{\infty} [m_{2k}(y)T_{2k}(t) + m_{2k+1}(y)T_{2k+1}(t)], \qquad (12)$$

$$f(y) = \sum_{k=1}^{\infty} [m_{2k}(y)f_{2k} + m_{2k+1}(y)f_{2k+1}], \qquad (13)$$

where, $f_k(t) = \langle f(y,t), w_k\rangle$*, along-with* $T_{2k}(t)$*,* $T_{2k+1}(t)$*,* g_{2k}*, and* g_{2k+1} *satisfy the fractional differential equations in the following system*

$${}^{BH}D^{(\vartheta,\chi)\zeta}_{0+,t}T_{2k}(t) = -\lambda_{1k}T_{2k}(t) + \Delta_k T_{2k+1}(t) + p(t)f_{2k}, \qquad (14)$$

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Online ISSN 3007-3197 Print ISSN 3007-3189
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$$^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}T_{2k+1}(t) = -\lambda_{2k}T_{2k+1}(t) + p(t)f_{2k+1} \qquad (15)$$

use remark (3.1) to get the solution of above differential equation

$$\begin{aligned} T_{2k}(t) &= e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{1k})\phi_{2k} \\ &+\Delta_k T_{2k+1} * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k}) \\ &+p(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k})f_{2k}, \qquad (16) \end{aligned}$$

$$\begin{aligned} T_{2k+1}(t) &= e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{2k})\phi_{2k+1} \\ &+p(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k})f_{2k+1}, \qquad (17) \end{aligned}$$

where $*$ represents Laplace convolution. Hence the solution of (12) becomes

$$\begin{aligned} z(y,t) = \sum_{k=1}^{\infty} &[m_{2k}(y)(e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{1k})\phi_{2k} \\ &+\Delta_k(e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{2k})\phi_{2k+1} \\ &+p(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k})f_{2k+1}) \\ &\times 1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k}) \\ &+p(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k})f_{2k}) + m_{2k+1}(y) \\ &\times (e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{2k})\phi_{2k+1} \\ &+p(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k})f_{2k+1})] \qquad (18) \end{aligned}$$

Now by applying bi-ordinal Hilfer Fractional Derivative on over-determination condition,

$$\int_0^{\pi} ({}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}z(y,t))d(y) = {}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}E(t), \qquad (19)$$

we obtained time-dependent source term, in the form of

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Volume 4, Issue 3 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189
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$$p(t) = [\int_0^{\pi} f(y)dy]^{-1}[{}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta} E(t) + \tau(t) + \int_0^{t} K(t,\tau)a(\tau)d\tau]. \quad (20)$$

where,

$$\int_0^{\pi} z_{yy}(y,t)dy = -\sum_{k=1}^{\infty} [\frac{2\lambda_{1k}}{(2k+1)}(e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{1k})\phi_{2k}$$
$$+\Delta_k[e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{2k})\phi_{2k+1}$$
$$+p(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k})f_{2k+1}]$$
$$1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k})$$
$$+p(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k})f_{2k})$$
$$-\left(\frac{\lambda_{2k}}{2\delta_y\alpha_k} + \frac{\lambda_{2k}}{\delta_y(2k+1)} + \frac{2\Delta_k}{2k+1}\right)$$
$$(e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{2k})\phi_{2k}$$
$$+p(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k})f_{2k+1})],$$

$$K(t,T) = -\sum_{k=1}^{\infty} [\frac{2\lambda_{1k}}{(2k+1)}[\Delta_k(1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(\tau-t;\lambda_{2k})f_{2k+1})]$$
$$\times (1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(\tau-t;-\lambda_{1k})) + [\frac{2\lambda_{1k}}{2k+1}$$
$$\times \left(1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(\tau-t;-\lambda_{1k})\right) f_2k$$
$$+(\frac{\lambda_{2k}}{2\delta_y\alpha_k} - \frac{\lambda_{2k}}{\delta_y(2k+1)} - \frac{2\Delta_k}{2k+1})(1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}$$
$$(\tau-t;-\lambda_{2k})f_{2k+1})], \quad (21)$$

and

$$\tau(t) = -\sum_{k=1}^{\infty} [\frac{2\lambda_{1k}}{(2k+1)}\phi_{2k} e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{1k})$$
$$+\frac{2\lambda_{1k}}{(2k+1)}\Delta_k \times e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{2k})\phi_{2k+1}$$
$$(1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k}))$$
$$-\left(\frac{\lambda_{2k}}{2\delta_y\alpha_k} - \frac{\lambda_{2k}}{\delta_y(2k+1)} - \frac{2\Delta_k}{2k+1}\right)$$
$$(e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{1k})\phi_{2k+1})]. \quad (22)$$

Existence of the solution

Consider the mapping $\mathcal{B}$ from

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Volume 4, Issue 3 (2026)

Online ISSN 3007-3197 Print ISSN 3007-3189

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$$\mathcal{B}: C([0,T]) \to C([0,T]), \qquad (23)$$

define by an operator

$$\mathcal{B}(p(t)) = p(t).$$

where $p(t)$ is given in the equation (20), with the norm

$$\| g \| := \max_{0\le t\le T} |g(t)|$$

where the operator $\mathcal{B}$ is

$$\mathcal{B}(p(t)) = [\int_0^{\pi} f(y)dy]^{-1}[{}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta} E(t) + \tau(t) + \int_0^t K(t,\tau)a(\tau)d\tau]$$

to demonistrate that $\mathcal{B}: C([0,T]) \to C([0,T])$ is well defined, we need to show that $p(t) \in C([0,T])$, $\mathcal{B}(p(t))$ is a continuous function.

Prove $T(t)$ and $K(t,\tau)$ is uniformly convergent by using Lemma 3.1, 3.2, 4.1, 4.2, together with the relation (11) which leads to the following relation for $\tau(t)$

$$t^2|\tau(t)| \le \sum_{k=1}^{\infty} [\frac{2|\lambda_{1k}|C_1}{(2k+1)k^2}(\frac{|\langle \phi''_{2k}(y), q_{2k}(y)\rangle|}{|k^2|})T^{\zeta(1-\vartheta)} + \frac{14|\lambda_{1k}|C_1^2}{(2k+1)k^3}T^{\zeta(1-\vartheta)}$$
$$\times \left(\frac{\left|\langle \phi^{(4)}_{2k+1}(y), q_{2k+1}(y)\rangle\right|}{|k^4|}\right) + \left(\frac{|\lambda_{2k}|}{2\delta_y \alpha_k} + \frac{|\lambda_{2k}|}{\delta_y(2k+1)} + \frac{14k}{2k+1}\right)$$
$$\times (T^{\zeta(1-\vartheta)-2}\frac{C_1}{k^2}(\frac{|\langle \phi^{(4)}_{2k+1}(y), q_{2k+1}(y)\rangle|}{|k^4|}))],$$

same procedure is followed for $|K(t,\tau)|$, eq (22) becomes

$$K(t,T) \le \sum_{k=1}^{\infty} [\frac{14|\lambda_{1k}|C_1^2}{(2k+1)k^3}\frac{|\langle f^{(4)}_{2k+1}, q_{2k+1}(y)\rangle|}{|k^4|} + \frac{2|\lambda_{1k}|C_1}{k^2(2k+1)}\frac{|\langle f''_{2k}, q_{2k}(y)\rangle|}{|k^2|}$$
$$+(\frac{|\lambda_{2k}|}{2\delta_y\alpha_k} + \frac{|\lambda_{2k}|}{\delta_y(2k+1)} + \frac{14k}{2k+1})\frac{C_1}{k^2}\frac{|\langle f^{(4)}_{2k+1}, q_{2k+1}(y)\rangle|}{|k^4|}]. \qquad (24)$$

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Cauchy Schwarz inequality, guarantee that the above inequalities are uniformly convergent. Since $\tau(t)$ and $|K(t,\tau)|$ are uniformly convergent. So, we may write it as

$$\| K(t,\tau) \| \leq K_1, \qquad (25)$$

and

$$0 \leq \frac{1}{M_1} \leq \int_0^\pi f(y)dy, \qquad (26)$$

to indicate that mapping (23) is contraction.

Let,

$$|\mathcal{B}(p(t)) - \mathcal{B}(b(t))| \leq [\int_0^\pi f(y)dy]^{-1}[\int_0^t (|p(t) - p'(t)K(t,\tau))d\tau|],$$

this result is supported by inequality (26) and (27),

$$|\mathcal{B}(p(t)) - \mathcal{B}(p'(t))| \leq M_1 K(t,\tau) \max_{0\leq t\leq T} |p(t) - p'(t)| \int_0^t d\tau,$$

$$\| \mathcal{B}(p(t)) - \mathcal{B}(p'(t)) \|_t \leq T M_1 K_1 \| p - p' \|, \qquad (27)$$

by Banach contraction mapping $\mathcal{B}(.)$ is contraction for $T(\frac{1}{M_1 K_1})$,
where $K_1 = \| K(t,\tau) \|$.

Now insures continuity results of $z(y,t)$, $z_{yy}(y,t)$ and ${}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta} u(y.t)$.

Firstly, we will check continuity for $z(y,t)$, for this purpose use Lemma 3.1, 3.2 we obtained

$$\sum_{k=1}^{\infty} t^2 |T_{2k}(t)| \leq \sum_{k=1}^{\infty} |T^{\zeta(1-\vartheta)}| C_1 C_3 \frac{\| \phi'' \|}{k^2} + 7\frac{C_1 C_3}{k} |T^{\zeta(1-\vartheta)}| \frac{\| \phi^{(4)} \|}{k^4}$$

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Online ISSN 3007-3197 Print ISSN 3007-3189
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$$+7\frac{C_1^2C_3C_4}{k^3}T^2\frac{\| f^{(4)} \|}{k^4}+\frac{C_1C_2C_3}{k^2}T^2\frac{\| f'' \|}{k^2},$$

$$\sum_{k=1}^{\infty} t^2|T_{2k+1}(t)| \leq \sum_{k=1}^{\infty} |T^{\zeta(1-\vartheta)}|\frac{C_1C_3}{k^2}|\frac{\| \phi^{(4)} \|}{k^4}+T^2\frac{C_1C_3C_4}{k^2}\frac{\| f^{(4)} \|}{k^4},$$

where, $\| q_k \| \leq C_3$, $\| p(t) \| \leq C_4$ *and* $\| t^2 \| \leq T^2$. *We already show that the uniform convergence of* $\sum_{k=1}^{\infty} |\phi_{2k}|, \sum_{k=1}^{\infty} |\phi_{2k+1}|, \sum_{k=1}^{\infty} |f_{2k}|$, *and* $\sum_{k=1}^{\infty} |f_{2k+1}|$, *so* $z(y,t)$ *represents a continuous function via Weierstrass M-test.*

By using same strategy for $z_{yy}(y,t)$, *we have*

$$t^2|z_{yy}(y,t)| \leq \sum_{k=1}^{\infty} [T^{\zeta(1-\vartheta)}C_1C_3\frac{\| \phi_{,,} \|}{k^2}+T^{\zeta(1-\vartheta)}C_3\frac{\| \phi^{(4)} \|}{k^4}(7k+C_1+\frac{7C_1}{k})$$
$$+T^2C_3C_4(C_1\frac{\|f^{(4)}\|}{k^4}(\frac{14+k}{k})+\frac{\|f''\|}{k^2})],$$

using Lemma 4.1 and 4.2, we can see that the series on the right-hand side are uniformly convergent. Consequently, Weierstrass M-test guarantee $z_{yy}(y,t)$ *is a continuous function.*

Finally, we will show the continuity result for ${}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}z(y,t)$, *with the help of the Lemma 4.1, and 4.2, we can write*

$$t^2|{}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}T_{2k}(t)| \leq T^{\zeta(1-\vartheta)}\frac{C_1}{k^4}\| \phi'' \| +T^{\zeta(1-\vartheta)}\frac{7C_1}{k^5}\| \phi^{(4)} \| (C_1+1)$$
$$+T^2\frac{C_1C_4}{k^6}\| f^{(4)} \| (7kC_1+1)+T^2\frac{C_4}{k^2}\| f'' \| (C_1+1),$$

$$t^2|{}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}T_{2k+1}(t)| \leq T^{\zeta(1-\vartheta)}C_1\frac{\| \phi^{(4)} \|}{k^4}+T^2C_4|\frac{\| f^{(4)} \|}{k^4}(C_1+1).$$

By Weierstrass M-test, ${}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}z(y,t)$ *is continuous functions.*

Uniqueness of the solution

Assume two solution sets $\{z_1(y,t), f_1(y)\}$ *and* $\{z_2(y,t), f_2(y)\}$, *then* $\bar{z}_1(y,t)=z_1(y,t)-z_2(y,t)$ *and* $\bar{f}(y)=f_1(y)-f_2(y)$

satisfies

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$$^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}\bar{z}(y,t) = \bar{z}_{yy}(y,t),$$

subject to the initial condition

$$I_{0+,t}^{(1-\zeta)(1-\chi)}\bar{z}(y,t)|_{t=0} = 0,$$

and the boundary measurements

$$\bar{z}(0,t) = 0, \bar{z}_y(0,t) + \bar{z}_y(\pi,t) + \rho\bar{z}(\pi,t) = 0, t \in (0,T].$$

Consider,

$$\bar{T}_k = \langle u_i(y,t), w_k(y)\rangle, \quad \bar{f}_k = \langle f_{ik}(y), w_k(y)\rangle,$$

apply bi-ordinal Hilfer fractional derivative

$$^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}\bar{T}_{2k} = \langle ^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}\bar{z}(y,t), q_{2k}(y)\rangle,$$

$$^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}\bar{T}_{2k+1} = \langle ^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}\bar{z}(y,t), q_{2k+1}(y)\rangle,$$

from these equations, the system has a differential equation of the form

$$^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}\bar{T}_{2k}(t) - \lambda_{1k}\bar{T}_{2k}(t) = 0,$$

$$^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}\bar{T}_{2k+1}(t) - \lambda_{2k}\bar{T}_{2k+1}(t) = 0,$$

following are the results of applying the Laplace transform on differential equations:

$$\bar{T}_{2k}(t) = e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t; -\lambda_{1k})\bar{\phi}_{2k},$$

$$\bar{T}_{2k+1}(t) = e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t; -\lambda_{2k})\bar{\phi}_{2k+1},$$

with the use of the initial condition, the above results become

$$\bar{T}_{2k}(0) = 0, \quad \bar{T}_{2k+1}(0) = 0,$$

which implies

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$$\bar{z}(y,t) = 0,$$

hence,

$$z(y,t) = f(y,t).$$

which proves given result is unique.

Theorem 5.2 *Under the following assumptions, the backward source problem (1)-(3) possesses a regular solution*

- $\psi \in C^4(0,\pi)$ *and* $\psi^{(n)}(0) = 0 = \psi^{(n)}(\phi), n = 0,1,2,3,$
- $g \in C^{2,0}_{y,t}(\Phi)$ *and* $\frac{\partial^m g}{\partial y^m}(0,t) = 0 = \frac{\partial^m g}{\partial y^m}(\pi,t), m = 0,1,$ *then their exists a unique solution.*

Proof. ***Construction of the solution:*** *As we consider spectral problem, the solution representation of the backward problem can be written as*

$$z(y,t) = \sum_{k=1}^{\infty} [m_{2k}(y)T_{2k}(t) + m_{2k+1}(y)T_{2k+1}(t)], \quad (28)$$

$$f(y) = \sum_{k=1}^{\infty} [m_{2k}(y)f_{2k} + m_{2k+1}(y)f_{2k+1}], \quad (29)$$

$$g(y,t) = \sum_{k=1}^{\infty} [m_{2k}(y)g_{2k}(t) + m_{2k+1}(y)g_{2k+1}(t)], \quad (30)$$

by talking inner product the differential forms of the given system is

$$\begin{aligned} {}^{BH}D^{(\vartheta,\chi)\zeta}_{0+,t} T_{2k}(t) &= -\lambda_{1k}T_{2k}(t) + \Delta_k T_{2k+1}(t) + A_1 f_{2k}\, g_{2k}(t) \\ &+ A_2 f_{2k}\, g_{2k+1}(t) + A_2 f_{2k+1}\, g_{2k}(t), \quad (31) \end{aligned}$$

$$\begin{aligned} {}^{BH}D^{(\vartheta,\chi)\zeta}_{0+,t} T_{2k+1}(t) &= -\lambda_{2k}T_{2k+1}(t) + A_3 f_{2k}\, g_{2k+1}(t) + A_3 f_{2k+1}\, g_{2k}(t) \\ &+ A_4 f_{2k+1}\, g_{2k+1}(t), \quad (32) \end{aligned}$$

where,

$$\langle m^2_{2k}(y), q_{2k}(y)\rangle = \frac{1}{6k+3}\left[\frac{8\alpha_k^4(8\alpha_k^2 - 6\vartheta^2) + \vartheta^4(12\alpha_k^2 - \vartheta^2)}{8\alpha_k^4(8\alpha_k^2 + 6\vartheta^2) + \vartheta^4(12\alpha_k^2 + \vartheta^2)} + 3\left(\frac{4\alpha_k^2 - \vartheta^2}{4\alpha_k^2 + \vartheta^2}\right)\right.$$

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Online ISSN 3007-3197 Print ISSN 3007-3189
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$$-2]+\frac{64\alpha_k^2\vartheta^2}{3\vartheta(8\alpha_k^4(8\alpha_k^2+6\vartheta^2)+\vartheta^4(12\alpha_k^2+\vartheta^2))}+[\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2}$$
$$\times(2\alpha_k^2+8k+2)]\frac{C_{2k}}{8\alpha_k(\alpha_k-2k-1)(4\alpha_k+2k+1)}$$
$$+\frac{C_{2k}}{4\vartheta(\alpha_k^2-4k^2-4k-1)}[(-\frac{4\alpha_k\vartheta}{4\alpha_k^3+\vartheta^2})(16k+2)]=A_1,$$

$$\langle m_{2k}(y)m_{2k+1}(y),q_{2k}(y)\rangle$$
$$=\frac{1}{2\delta_k}\{\frac{1}{8\alpha_k(\alpha_k-2k-1)(\alpha_k+2k+1)}[\alpha_k(\alpha_k-2k-1)$$
$$\times(\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2})+\alpha_k(\alpha_k+2k+1)(\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2})-2(\alpha_k$$
$$-2k-1)(\alpha_k+2k+1)(\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2})-8k^2-8k-2]$$
$$-\frac{2k+1}{8\alpha_k(\alpha_k+2k+1)}\left(\frac{4\alpha_k\vartheta}{4\alpha_k^2+\vartheta^2}\right)+\frac{2k+1}{8\alpha_k(\alpha_k-2k-1)}$$
$$\times(\frac{4\alpha_k\vartheta}{4\alpha_k^2+\vartheta^2})-\frac{C_{2k}}{4(2k+1)(4\alpha_k-2k-1)(4\alpha_k+2k+1)}$$
$$\times[(2k+1)(4\alpha_k-2k-1)(\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2})-(2k+1)$$
$$(4\alpha_k+2k+1)(\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2})-32[(\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2})+1]\alpha_k^2$$
$$+2\left(2k+1)^2\times\left(\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2}\right)\right]-\frac{C_{2k}\alpha_k}{2\alpha_k(4\alpha_k+2k+1)}$$
$$(\frac{4\alpha_k\vartheta}{4\alpha_k^2+\vartheta^2})+\frac{C_{2k}\alpha_k}{2\alpha_k(4\alpha_k-2k-1)}(\frac{4\alpha_k\vartheta}{4\alpha_k^2+\vartheta^2})\}=A_2,$$

$$\langle m_{2k}(y)m_{2k+1}(y),q_{2k+1}(y)\rangle=\frac{C_{2k}}{4(2k+1)(4\alpha_k-2k-1)(4\alpha_k+2k+1)}$$
$$\times[-(2k+1)(4\alpha_k-2k-1)(\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2})$$
$$-(2k+1)(4\alpha_k+2k+1)(\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2})-32$$
$$\times[(\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2})+1]\alpha_k^2+2(2k+1)^2$$
$$\times(\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2})]+\frac{C_{2k}\alpha_k}{2\vartheta(4\alpha_k+2k+1)}(\frac{4\alpha_k\vartheta}{4\alpha_k^2+\vartheta^2})$$
$$-\frac{C_{2k}\alpha_k}{2\vartheta((4\alpha_k-2k-1))}(\frac{4\alpha_k\vartheta}{4\alpha_k^2+\vartheta^2})$$
$$+\frac{C_{2k}}{8\alpha_k(\alpha_k-2k-1)(\alpha_k+2k+1)}[-\alpha_k(\alpha_k$$

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Online ISSN 3007-3197 Print ISSN 3007-3189
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$$-2k-1)\times\left(\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2}\right)-\alpha_k(\alpha_k+2k+1)$$
$$\times\left(\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2}\right)+2(\alpha_k-2k-1)(\alpha_k+2k+1)$$
$$\times\left(\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2}\right)+8k^2+8k+2]+\frac{C_{2k}\alpha_k}{4\vartheta(\alpha_k+2k+1)}$$
$$\times\left(\frac{4\alpha_k\vartheta}{4\alpha_k^2+\vartheta^2}\right)-\frac{C_{2k}\alpha_k}{4\vartheta(\alpha_k-2k-1)}\left(\frac{4\alpha_k\vartheta}{4\alpha_k^2+\vartheta^2}\right)$$
$$+\frac{C_{2k}}{2\vartheta}\left(\frac{4\alpha_k\vartheta}{4\alpha_k^2+\vartheta^2}\right)=A_3,$$

$$\langle m_{2k+1}^2(y),q_{2k+1}(y)\rangle=\frac{C_{2k}}{12\delta_k\alpha_k}[(\frac{8\alpha_k^4(8\alpha_k^2-6\vartheta^2)+\vartheta^4(12\alpha_k^2-\vartheta^2)}{8\alpha_k^4(8\alpha_k^2+6\vartheta^2)+\vartheta^4(12\alpha_k^2+\vartheta^2)})$$
$$-\frac{C_{2k}}{4\delta_k\alpha_k}\left(\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2}\right)+\frac{1}{6\delta_k\alpha_k}]-\frac{C_{2k}}{6\delta_k\vartheta}$$
$$\times\left(\frac{64\alpha_k^2\vartheta^2}{8\alpha_k^4(8\alpha_k^2+6\vartheta^2)+\vartheta^4(12\alpha_k^2+\vartheta^2)}\right)$$
$$+\frac{C_{2k}}{8\alpha_k^4(-32k^2-32k-8)}[-(\alpha_k^2+(-2k-1)\alpha_k)$$
$$\times\left(\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2}\right)+(\alpha_k^2+(+2k+1)\alpha_k)\left(\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2}\right)$$
$$+(-2\alpha_k^2+8k^2+8k+2)\left(\frac{4\alpha_k^2-\vartheta^2}{4\alpha_k^2+\vartheta^2}\right)$$
$$-8k^2-8k-2]-\frac{C_{2k}\alpha_k}{4\vartheta(\alpha_k+2k+1)}\left(\frac{4\alpha_k\vartheta}{4\alpha_k^2+\vartheta^2}\right)$$
$$+\frac{C_{2k}\alpha_k}{4\vartheta(\alpha_k-2k-1)}\left(\frac{4\alpha_k\vartheta}{4\alpha_k^2+\vartheta^2}\right)+\frac{C_{2k}}{2\vartheta}\left(\frac{4\alpha_k\vartheta}{4\alpha_k^2+\vartheta^2}\right)$$
$$=A_4.$$

Using Laplace transform, we get values of $T_{2k}(t)$ and $T_{2k+1}(t)$ as

$$\begin{aligned}T_{2k}(t)&=e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{1k})\phi_{2k}+\Delta_k T_{2k+1}(t)\\&\times 1*e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k})+g_{2k}(t)\\&\times 1*e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k})A_1f_{2k}+g_{2k+1}(t)\\&\times 1*e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k})A_2f_{2k}+g_{2k}(t)\\&\times 1*e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k})A_2f_{2k+1},\qquad(33)\end{aligned}$$

$$\begin{aligned}T_{2k+1}(t)&=e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{2k})\phi_{2k+1}+g_{2k+1}(t)\\&\times 1*e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k})A_3f_{2k}+g_{2k}(t)\\&\times 1*e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k})A_3f_{2k+1}+g_{2k+1}(t)\\&\times 1*e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k})A_4f_{2k+1},\qquad(34)\end{aligned}$$

temperature over-determination condition is required to get f_{2k} and f_{2k+1}, which gives

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Online ISSN 3007-3197 Print ISSN 3007-3189
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$$f_{2k} = (\frac{\xi_1}{\xi_2}), \qquad (35)$$

and

$$f_{2k+1} = (\frac{\xi_3}{\xi_4}), \qquad (36)$$

where,

$$\begin{aligned}\xi_1 = {} & [\psi_{2k}(y) - e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{1k})\phi_{2k} \\ & -\Delta_k\psi_{2k+1}(y)(1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k}))] \\ & \times [A_3 g_{2k}(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k}) \\ & +A_4 g_{2k+1}(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k})] \\ & +[A_2\psi_{2k+1}(y) - A_2 e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{2k})\phi_{2k+1}] \\ & \times (g_{2k}(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k})),\end{aligned}$$

$$\begin{aligned}\xi_2 = {} & 1 - [A_2A_3 g_{2k+1}(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k})] \\ & \times [g_{2k}(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k})] \\ & \times [A_3 g_{2k}(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k}) \\ & +A_4 g_{2k}(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k})],\end{aligned}$$

$$\begin{aligned}\xi_3 = {} & [A_1 g_{2k}(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k}) \\ & +A_2 g_{2k+1}(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k})] \\ & \times [\psi_{2k+1}(y) - e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{2k})\phi_{2k+1} \\ & +A_3\psi_{2k}(y) - A_3 e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{1k})\phi_{2k} \\ & -\Delta_k A_3\psi_{2k+1}(y)(1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k}))] \\ & \times g_{2k}(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k}),\end{aligned}$$

$$\begin{aligned}\xi_4 = {} & 1 - [A_1A_3 g_{2k}(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k}) \\ & +A_2 g_{2k}(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k})] \\ & \times [A_2 g_{2k}(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k}) \\ & \times (g_{2k}(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k}))],\end{aligned}$$

so the solution of backward source problem is

$$\begin{aligned}z(y,t) = {} & \sum_{k=1}^{\infty}\{sin((2k+1)(y))[e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{1k})\phi_{2k} \\ & +\Delta_k(e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{2k})\phi_{2k+1} + g_{2k+1}(t) \\ & \times 1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k})A_3(\frac{\xi_1}{\xi_2}) + g_{2k}(t) \\ & \times 1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k})A_3(\frac{\xi_3}{\xi_4}) + g_{2k+1}(t) \\ & \times 1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k})A_4(\frac{\xi_3}{\xi_4}))\end{aligned}$$

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Online ISSN 3007-3197 Print ISSN 3007-3189
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$$\times (1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k})) + g_{2k}(t)$$
$$\times 1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k})A_1(\frac{\xi_1}{\xi_2}) + g_{2k+1}(t)$$
$$\times 1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k})A_2(\frac{\xi_1}{\xi_2}) + g_{2k}(t)$$
$$\times 1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{1k})A_2(\frac{\xi_3}{\xi_4})]$$
$$+(\frac{(sin(2\alpha_k(y)) - sin((2k+1)(y)))}{2\delta_k})(e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{2k})$$
$$\times \phi_{2k+1} + g_{2k+1}(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k})A_3(\frac{\xi_1}{\xi_2})$$
$$+g_{2k}(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k})A_3(\frac{\xi_3}{\xi_4}) + g_{2k+1}(t)$$
$$* e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k})A_4(\frac{\xi_3}{\xi_4}))\}. \qquad (37)$$

Existence of the solution

Under certain regularity conditions on $\phi(y)$ and $f(y)$, we will now present the existence results, for this $z(y,t)$ must be shown to be regular, and this can be done by demonstrating that

- $z(y,t)$
- ${}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta} z(y,t)$,
- $z_{yy}(y,t)$

is continuous function.

By utilizing equation (36), (37) along with the Lemma 3.1 and 3.2, we have

$$t^2|f_{2k+1}| \le \frac{\xi_5}{\xi_2},$$

$$t^2|f_{2k}| \le \frac{\xi_6}{\xi_4},$$

where,

$$\xi_5 = [\frac{C_1^2 C_3}{k^4} 2 \parallel g(A_1 + A_2)][\frac{\parallel \psi^{(4)} \parallel}{k^4} C_3 T^2 (1 + \frac{7 A_3 C_1}{k})$$
$$+T^{\zeta(1-\vartheta)} \frac{C_1 C_3}{k^2} (\frac{\parallel \phi^{(4)} \parallel}{k^4} + \frac{\parallel \phi'' \parallel}{k^2} A_3) + \frac{T^2 A_3 C_3}{k^2} \parallel \psi'' \parallel],$$

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$$\xi_6=\frac{C_1^2C_3^2}{k^4}(A_3C_3^2\parallel g\parallel^2)(T^2k^2C_3\frac{\parallel\psi''\parallel}{k^2}+C_1C_3T^{\zeta(1-\vartheta)}(\frac{\parallel\phi''\parallel}{k^2}+\frac{A_2}{k^2}\frac{\parallel\psi^{(4)}\parallel}{k^4})$$
$$+T^2C_3\frac{\parallel\psi^{(4)}\parallel}{k^4}(\frac{7C_1}{k}+A_2)),$$

for solving ξ_2, and ξ_4 use Lemma 3.1, 3.2 together with The Lemma 4.1 and 4.2, it will be

$$1-\frac{C_1^3C_3}{k^6}\parallel g\parallel^3(A_1A_2A_3+A_2^2)\geq 1-\frac{C_1^3C_3}{k^6}\parallel g\parallel^3,$$

$$1-A_2A_3\frac{C_1^3C_3^2}{k^6}\parallel g\parallel^3(A_3+A_4)\geq 1-\frac{C_1^3C_3^2}{k^6}\parallel g\parallel^3,$$

this implies that the denominator will be non-zero, as

$$\langle m_{2k}^2(y),q_{2k}(y)\rangle<1,$$

$$\langle m_{2k}(y)m_{2k+1}(y),q_{2k}(y)\rangle<1,$$

$$\langle m_{2k}(y)m_{2k+1}(y),q_{2k+1}(y)\rangle<1,$$

$$\langle m_{2k+1}^2(y),q_{2k+1}(y)\rangle<1,$$

so

$$\frac{1}{\xi_2}\leq 1,$$

and

$$\frac{1}{\xi_4}\leq 1.$$

Hence, the estimates will be

$$t^2|f_{2k+1}|\leq\xi_5, \qquad (38)$$

$$t^2|f_{2k}|\leq\xi_6. \qquad (39)$$

We can determine that $f(y)$ is a continuous function by using Equations (38) and (39) alongside Weierstrass M-test.

To establish the continuity of the function $z(y,t)$, we will illustrate that the functions

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Online ISSN 3007-3197 Print ISSN 3007-3189
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T_{2k} and T_{2k+1} are continuous by using Lemma 3.1, 3.2 4.1, 4.2 and relation (11) together with Cauchy-Bunyakovsky-Schwarz inequality we have the following estimates for $|T_{2k}(t)|$ and $|T_{2k+1}(t)|$

$$\sum_{k=1}^{\infty} t^2|T_{2k}(t)| \leq \sum_{k=1}^{\infty} \frac{C_1}{k^4} \parallel \phi'' \parallel T^{\zeta(1-\vartheta)} + \frac{7C_1^2}{k^3}[T^{\zeta(1-\vartheta)}\frac{\parallel \phi|^{(4)} \parallel}{k^4} + \parallel g$$
$$\parallel C_3(A_3\frac{\parallel f'' \parallel}{k^2})] + \frac{C_3 \parallel f^{(4)} \parallel}{k^4}(A_3 \parallel g \parallel + A_4 \parallel g \parallel)$$
$$+\frac{\parallel f'' \parallel}{k^2}\frac{C_1C_3}{k^2}(A_1 \parallel g \parallel + A_2 \parallel g \parallel)$$
$$+A_2\frac{\parallel f^{(4)} \parallel}{k^4}\frac{C_1C_3}{k^2} \parallel g \parallel, \qquad (40)$$

$$\sum_{k=1}^{\infty} t^2|T_{2k+1}(t)| \leq \sum_{k=1}^{\infty} [\frac{C_1}{k^2}(\frac{\parallel \phi^{(4)} \parallel}{k^4}T^{\zeta(1-\vartheta)} + A_3C_3T^2 \parallel g \parallel \frac{\parallel f'' \parallel}{k^2}$$
$$+A_3C_3T^2\frac{\parallel f^{(4)} \parallel}{k^4} \parallel g \parallel + A_4C_3\frac{\parallel f^{(4)} \parallel}{k^4} \parallel g \parallel)].$$

Weierstrass M-test guarantee the continuity of $z(y,t)$.
To prove regularity of the series expansion we will also determine the convergence of ${}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}z(y,t)$, and $z_{yy}(y,t)$, that is the corresponding series is uniformly convergent, Cauchy-Bunyakovsky-Schwarz inequality with Lemma 4.1 and 4.2, we attain

$$t^2|{}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}T_{2k}(t)| \leq T^{\zeta(1-\vartheta)}\frac{C_1C_3}{k^2}\frac{\parallel \phi'' \parallel}{k^2} + \frac{7C_1}{k}(T^{\zeta(1-\vartheta)}\frac{\parallel \phi^{(4)} \parallel}{k^4}$$
$$+2C_3 \parallel g \parallel T^2 \times (\frac{\parallel f'' \parallel}{k^2} + \frac{\parallel f^{(4)} \parallel}{k^4})),$$

Weierstrass M-test ensures that the function involve in $t^2|{}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}T_{2k}(t)|$ and $t^2|{}^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}T_{2k+1}(t)|$ is a continuous function. Now check continuity result for $z_{yy}(y,t)$

$$z_{yy}(y,t) = \sum_{k=1}^{\infty} [m_{2k}''(y)[e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{1k})\phi_{2k} + \Delta_k$$
$$\times (e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t;-\lambda_{2k})\phi_{2k+1} + A_3f_{2k}g_{2k+1}(t)$$
$$\times 1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k}) + A_3f_{2k+1}g_{2k}(t)$$
$$\times 1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k}) + A_4f_{2k+1}g_{2k+1}(t)$$
$$\times 1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t;-\lambda_{2k})) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}$$

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$$\begin{aligned}
&\times (t; -\lambda_{1k}) + A_1 f_{2k} g_{2k}(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t; -\lambda_{1k}) \\
&+ g_{2k+1}(t) * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t; -\lambda_{1k}) A_2 f_{2k} + g_{2k}(t) \\
&\times 1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t; -\lambda_{1k}) A_2 f_{2k+1}] + m_{2k+1}''(y) \\
&\times (e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t; -\lambda_{2k}) \phi_{2k+1} + g_{2k+1}(t) \\
&\times 1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t; -\lambda_{2k}) A_3 f_{2k} + g_{2k}(t) \\
&\times 1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t; -\lambda_{2k}) A_3 f_{2k+1} + g_{2k+1}(t) \\
&\times 1 * e_{\chi+\zeta(\vartheta-\chi),\chi+\zeta(\vartheta-\chi)}(t; -\lambda_{2k}) A_4 f_{2k+1})],
\end{aligned}$$

using Lemma 3.1, 3.2, 4.1 additionally4.2 we have the following form for $|z_{yy}(y,t)|$

$$\begin{aligned}
|z_{yy}(y,t)| \le \sum_{k=1}^{\infty} &[T^{\zeta(1-\vartheta)} C_1 \frac{\| \phi'' \|}{k^2} + 7k(\frac{C_1}{k^2} T^{\zeta(1-\vartheta)} \frac{\| \phi^{(4)} \|}{k^4}) + \frac{\| f'' \|}{k^2} \| g \| \\
&\times \frac{C_1 C_3}{k^2}(7k(C_1+1)+2) + \frac{\|f^{(4)}\|}{k^4} \| g \| \frac{C_1 C_3}{k^2}(C_1(7k + \frac{1}{k^2})) \\
&+ \frac{\|f^{(4)}\|}{k^4} \| g \| C_3(C_1(2+7k)+1) + (1+7k)(T^{\zeta(1-\vartheta)} C_1 \frac{\|\phi^{(4)}\|}{k^4})].
\end{aligned}$$

Cauchy-Bunyakovsky-Schwarz inequality proves that the sequence represented on the right-hand side are uniformly convergent. According to the Weierstrass M-test, this is a continuous function.

Uniqueness of the solution

Assume two regular solutions of the backward problem $\{z_1(y,t), f_1(y)\}$ *and* $\{z_2(y,t), f_2(y)\}$, *then* $\bar{z}_1(y,t) = z_1(y,t) - z_2(y,t)$ *and* $\bar{f}(y) = f_1(y) - f_2(y)$ *satisfies*

$$^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta} \bar{z}(y,t) = \bar{z}_{yy}(y,t),$$

subject to initial condition

$$I_{0+,t}^{(1-\zeta)(1-\chi)} \bar{z}(y,t)|_{t=0} = 0,$$

subject to Boundary condition

$$\bar{z}(0,t) = 0, \bar{z}_y(0,t) + \bar{z}_y(\pi,t) + \rho\bar{z}(\pi,t) = 0, t \in (0,T],$$

Consider,

$$\bar{T}_k = \langle u_i(y,t), w_k(y) \rangle, \quad \bar{f}_k = \langle f_{ik}(y), w_k(y) \rangle,$$

apply bi-ordinal Hilfer fractional derivative to get differential form

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$$^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}\bar{T}_{2k} = \langle ^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}\bar{z}(y,t), q_{2k}(y)\rangle,$$

$$^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}\bar{T}_{2k+1} = \langle ^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}\bar{z}(y,t), q_{2k+1}(y)\rangle,$$

from these equations the system have differential equation of the form

$$^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}\bar{T}_{2k}(t) + \lambda_{1k}\bar{T}_{2k}(t) = 0,$$

$$^{BH}D_{0+,t}^{(\vartheta,\chi)\zeta}\bar{T}_{2k+1}(t) + \lambda_{2k}\bar{T}_{2k+1}(t) = 0,$$

applying the Laplace transform and initial data yields the following differential solution

$$\bar{T}_{2k}(t) = e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t; -\lambda_{1k})\bar{\phi}_{2k},$$

$$\bar{T}_{2k+1}(t) = e_{\chi+\zeta(\vartheta-\chi),(1-\chi)(\zeta-1)}(t; -\lambda_{2k})\bar{\phi}_{2k+1},$$

by using initial condition, above results becomes

$$\bar{T}_{2k}(t) = 0, \quad \bar{T}_{2k+1}(t) = 0,$$

which implies

$$\bar{z}(y,t) = 0,$$

hence

$$z(y,t) = f(y,t).$$

which proves given result is unique.

References

[1] J. Knighton, K. Singh, J. Evaristo, Understanding catchment-scale forest root water uptake strategies across the continental united states through inverse ecohydrological modeling, Geophysical Research Letters 47 (1) (2020) e2019GL085937.
[2] P. Figueiredo, Development of an iterative method for solving multidimensional inverse heat conduction problems, Lehrstuhl für Wärme-und Stoffübertragung RWTH Aachen (2014).
[3] D. C. Forney, D. H. Rothman, Common structure in the heterogeneity of plant-matter decay, Journal of The Royal Society Interface 9 (74) (2012) 2255–2267.

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 3 (2026)

Online ISSN 3007-3197 Print ISSN 3007-3189

http://amresearchreview.com/index.php/Journal/about

[4] P. Tahmasebi, F. Javadpour, M. Sahimi, Stochastic shale permeability matching: Three-dimensional characterization and modeling, International Journal of Coal Geology 165 (2016) 231–242.
[5] B. Yilmaz, A new type electromagnetic curves in optical fiber and rotation of the polarization plane using fractional calculus, Optik 247 (2021) 168026.
[6] S. S. Ray, A. Atangana, S. Noutchie, M. Kurulay, N. Bildik, A. Kilicman, Fractional calculus and its applications in applied mathematics and other sciences (2014).
[7] V. K. Srivastava, S. Kumar, M. K. Awasthi, B. K. Singh, Two-dimensional time fractional-order biological population model and its analytical solution, Egyptian journal of basic and applied sciences 1 (1) (2014) 71–76.
[8] W. Islam, M. Younis, S. T. R. Rizvi, Optical solitons with time fractional nonlinear schrödinger equation and competing weakly nonlocal nonlinearity, Optik 130 (2017) 562–567.
[9] E. A. Gonzalez, I. Petráš, Advances in fractional calculus: Control and signal processing applications (2015).

[10] V. Yakhot, D. A. Donzis, Anomalous exponents in strong turbulence, Physica D: Nonlinear Phenomena 384 (2018) 12–17.
[11] I. M. Sokolov, J. Klafter, From diffusion to anomalous diffusion: a century after einstein's brownian motion, Chaos: An Interdisciplinary Journal of Nonlinear Science 15 (2) (2005) 026103.
[12] S. G. Samko, A. A. Kilbas, O. I. Marichev, et al., Fractional integrals and derivatives, Vol. 1, Gordon and breach science publishers, Yverdon Yverdon-les-Bains, Switzerland, 1993.
[13] L. Yang, J.-N. Yu, Z.-C. Deng, An inverse problem of identifying the co-efficient of parabolic equation, Applied Mathematical Modelling 32 (10) (2008) 1984–1995.
[14] Z.-C. Deng, L. Yang, J.-N. Yu, G.-W. Luo, Identifying the radiative coefficient of an evolutional type heat conduction equation by optimization method, Journal of mathematical analysis and applications 362 (1) (2010) 210–223.
[15] Z.-C. Deng, L. Yang, N. Chen, Uniqueness and stability of the minimizer for a binary functional arising in an inverse heat conduction problem, Journal of mathematical analysis and applications 382 (1) (2011) 474–486.
[16] N. H. Tuan, L. D. Long, V. T. Nguyen, T. Tran, On a final value Problem for the time-fractional diffusion equation with inhomogeneous source, Inverse Problems in Science and Engineering 25 (9) (2017) 1367–1395.
[17] M. F. Al-Jamal, A backward problem for the time-fractional diffusion equation, Mathematical Methods in the Applied Sciences 40 (7) (2017) 2466–2474.

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 3 (2026)

Online ISSN 3007-3197 Print ISSN 3007-3189
http://amresearchreview.com/index.php/Journal/about

[18] J. Wen, J. F. Cheng, The method of fundamental solution for the inverse source problem for the space-fractional diffusion equation, Inverse Problems in Science and Engineering 26 (2018) 925–941.
[19] M. Z. S. Tatar, R. Tinaztepe, Numerical solutions of direct and inverse problems for a time fractional viscoelastoplastic equation, Journal of Engineering Mechanics 143.
[20] S. Tatar, Monotonicity of input–output mapping related to inverse elastoplastic torsional problem, Applied Mathematical Modelling 37 (23) (2013) 9552–9561.
[21] A. Kawamoto, Lipschitz stability estimates in inverse source problems for a fractional diffusion equation of half order in time by carleman estimates, Journal of Inverse and Ill-posed Problems 26 (5) (2018) 647–672.
[22] P. Feng, E. T. Karimov, Inverse source problems for time-fractional mixed parabolic-hyperbolic-type equations, Journal of Inverse and Ill-posed Problems 23 (4) (2015) 339–353.
[23] J. Jia, J. Peng, J. Yang, Harnack's inequality for a space–time fractional diffusion equation and applications to an inverse source problem, Journal of Differential Equations 262 (8) (2017) 4415–4450.
[24] S. Tatar, S. Ulusoy, An inverse problem for a nonlinear diffusion equation with time-fractional derivative, Journal of Inverse and Ill-posed Problems 25 (2) (2017) 185–193.
[25] S. Tatar, S. Ulusoy, Analysis of direct and inverse problems for a fractional elastoplasticity model, Filomat 31 (3) (2017) 699–708.
[26] M. Sadybekov, G. Dildabek, A. Tengayeva, Constructing a basis from systems of eigenfunctions of one not strengthened regular boundary value problem, Filomat 31 (4) (2017) 981–987.
[27] E. Karimov, M. Ruzhansky, B. Toshtemirov, A boundary-value problem for a mixed type equation involving hyper-bessel fractional differential operator and hilfer's bi-ordinal fractional derivative, arXiv preprint arXiv:2103.08989 (2021).
[28] R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogosin, et al., Mittag-Leffler functions, related topics and applications, Springer, 2020.
[29] I. Podlubny, Mathematics in science and engineering-fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications (1999).
[30] N. Laskin, Some applications of the fractional poisson probability distribution, Journal of Mathematical Physics 50 (11) (2009) 113513.
[31] M. Ali, S. Aziz, Some inverse problems for time-fractional diffusion equation with nonlocal samarskii-ionkin type condition, Mathematical Methods in the Applied Sciences 44 (10) (2021) 8447–8462.