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Derivation Of Ordinary Generating Function And Consequent Orthogonality Property Of Efficient Orthogonal Boubaker Polynomials

Ayaz Ali Siyal
Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro, Pakistan

Muhammad Mujtaba Shaikh*
Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro, Pakistan

Sadia Arain
Department of College Education, Government of Sindh, Pakistan

Iqrar Ali Pali
Department of Mathematics and Statistics, Quaid-e-Awam University of Engineering, Science and Technology, 67480, Sindh-Pakistan

Sanaullah Dehraj
Department of Mathematics and Statistics, Quaid-e-Awam University of Engineering, Science and Technology, 67480, Sindh-Pakistan

Muhammad Hussain Baig
Department of Mathematics and Statistics, Quaid-e-Awam University of Engineering, Science and Technology, 67480, Sindh-Pakistan

ABSTRACT

In this paper, we focus the derivation of an unexplored property of recently introduced efficient orthogonal Boubaker polynomials (EOBPs) – the ordinary generating function (OGF). The derivation of an OGF for the EOBPs opens ways for investigating other classical properties and recurrence relations of the EOBPs. We first discuss the establishment of the OGF for EOBPs followed by its consequent aid in determining complete orthogonalization property of the EOBPs. The orthogonality property is derived based on the developed OGF here. These results can lead to several seminal developments utilizing the application of EOBPs in mathematical sciences and allied fields.

Introduction

The Boubaker polynomials (BPs) were first introduced in 2007 by a Tunisian scientist Prof. Dr. Ing. Karem Boubaker as a consequence of solving mathematical model in terms of heat equation [1]. Later, the polynomial was investigated further by the

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pioneer himself and other researchers and a number of mathematical properties of the BPs were developed. Along with classical orthogonal polynomials like Legendre and Chebyshev polynomials, Boubaker polynomials have appeared as an alternative family of polynomials with good prospects of numerical application. Boubaker polynomials have easy recursive property and have been effectively utilized in solving boundary value problems, heat transfer and integral equations. [2-8]. The BPs also spans a lot of literature in mathematical sciences in the past two decades which deals with treatment – analytical and numerical – of applied models in science and engineering, as reviewed in [9]. On the other hand, a few studies also focused on obtaining more efficient variants of the BPs focusing the orthogonality properties and simplicity of the polynomial expressions which are suited in different applications. For example, Boubaker-Turki polynomials [10], orthogonal Boubaker-Turki polynomials [7], modified orthogonal Boubaker polynomials [11-12], Ahmed's orthogonal variant of Boubaker polynomials [6] and efficient orthogonal Boubaker polynomial (EOBP) variant proposed recently by Siyal, Shaikh and Cesarano [13]. In the paper [13], authors exhaustively discussed the orthogonalization drive on the BPs in the past two decades with their critical appreciation. Further, authors suggested the EOBPs as a more realistic expression with regards to orthogonality, simplicity and nearness to the source of the conventional BPs.

There are namouras applications in which Boubaker polynomials have been utilized for few are mention. One such important application is found in viscoelastic material systems, where stress–strain relationships depend not only on the current state but also on the past deformation history [14]. A typical fractional dynamic model for a viscoelastic system can be expressed as:

$$ {}_{t_0}^{C}D_t^{\alpha} = A(t)x(t) + B(t)u(t) \quad (1)$$

where $0 < \alpha < 1$, $x(t)$denotes the state variable related to deformation, and $u(t)$ represents the control input. The aim is to determine an optimal control function $u(t)$ that minimizes a performance index of the for

$$J(u) = \int_0^t (x^2(t) + u^2(t))dt \quad (2)$$

subject to the fractional state equation. Such models have been widely used to capture memory-dependent mechanical behavior more realistically than classical control formulations [15].

Another important application of fractional optimal control appears in anomalous diffusion and transport processes, which are commonly encountered in environmental sciences, groundwater flow, and heat transfer phenomena. In such systems, the diffusion process may be governed by a fractional differential equation of the form:

$$ {}_{t_0}^{C}D_t^{\alpha}x(t) = k\frac{\partial^2 x(t)}{\partial x^2} + u(t), \quad (3)$$

where the fractional derivative accounts for sub-diffusive or super-diffusive behavior. The control input $u(t)$is introduced to influence the system dynamics in such a way that a prescribed performance measure is optimized. Fractional-order formulations of

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these models offer a more realistic representation of physical phenomena than classical diffusion-based equations, particularly in systems exhibiting memory effects [16]. Fractional-order dynamics are used in such applications, and they enhance stability analysis and modeling. Fractional optimal control is also a potent instrument of contemporary engineering because the optimal control framework would guarantee efficient performance of the system and the fulfillment of the physical constraints [17].

These fractional optimal control models are often too mathematically complicated to allow analytical solutions. Thus, orthogonal polynomial approximations of numerical techniques become crucial. Fractional differential constraints can be solved to systems of algebraic equations by doing a polynomial expansion of the state and control variables and using an operational set of matrices. This transformation allows effective numerical solution schemes and encourages the application of Fourier based spectral schemes based on Boubaker polynomials to solve FOCPs.

In this paper, we derive ordinary generating function (OGF) for the EOBPs developed in the study [13]. Further, using the derived OGF of the EOBPs, we also give an alternate proof to the complete orthogonality relation for the EOBPs. The contributions are discussed theoretically in form of proved theorems in this paper.

Material and methods

The conventional BPs [1] are defined as a series solution of the following characteristic differential equation:

$$(x^2-1)(3nx^2+n-2)\frac{d^2y}{dx^2}+3n(nx^2+3n-2)\frac{dy}{dx} -n(3n^2x^2+n^2-6n+8)y=0. \quad (4)$$

The compact form of the Boubaker polynomials is expressed in(5):

$$B_n(x)=\sum_{p=0}^{\frac{n}{2}}\frac{n-4p}{n-p}\binom{n-p}{p}(-1)^p X^{(n-2p)}. \quad (5)$$

The recurrence relation of these polynomials is of three terms, as in (6):

$$B_m(x)=x\,B_{m-1}(x)-B_{m-2}(x), \quad \text{for } m>2. \quad (6)$$

Through (4)-(6), we can list first few Boubaker polynomials, which are:

$B_0(x)=1,$
$B_1(x)=x,$
$B_2(x)=x^2+2,$
$B_3(x)=x^3+x,$
$B_4(x)=x^4-2,$
$B_5(x)=x^5-x^3-3x,$
$B_6(x)=x^6-2x^4-3x^2+2,$
$B_7(x)=x^7-3x^5-x^3+5x.$

The generating function for BPs is given in (7) as:

$$G(x,t)=\frac{1+3t^2}{1+t(t-x)}. \quad (7)$$

Equation (7) gives the OGF for the conventional BPs. These polynomials are not

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orthogonal. The known generating functions and orthogonality relations of well-known Hermite, Leguerre and Legendre polynomials are shown in Table 1.

Table 1. Basic properties of some orthogonal polynomials

Polynomials	Generating function	Orthogonality relation
Hermite	$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = e^{2xt-t^2}$	$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)dx = \begin{cases} 0 & m \neq n \\ 2^n n! \sqrt{\pi} & m = n \end{cases}$
Leguerre	$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x) = \frac{1}{(1-t)} e^{\frac{-xt}{1-t}}$	$\int_{0}^{\infty} e^{-x} L_m(x) L_n(x)dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$
Legendre	$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$	$\int_{-1}^{1} P_m(x) P_n(x)dx = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases}$

The generating functions of Hermite and Leguerre polynomials in Table 1 are in exponential form, whereas for the Legendre polynomials is in ordinary form, similar to the OGF of the Boubaker polynomials given in (7).

The EOBPs [13] correspond to the Strum-Liouville form:

$$\frac{1}{\sqrt{4-x^2}} \frac{d}{dx}\left[(4-x^2)^{\frac{3}{2}} \frac{dy}{dx}\right] + \lambda_n y = 0. \quad (8)$$

The EOBPs are orthogonal in $[-2,2]$ subject to the weight function [13]:

$$w(x) = \frac{1}{2}\sqrt{4-x^2}. \quad (9)$$

The starting few terms of EOBPs are:

$Q_0(x) = 1,$
$Q_1(x) = x,$
$Q_2(x) = x^2 - 1,$
$Q_3(x) = x^3 - 2x,$
$Q_4(x) = x^4 - 3x^2 + 1,$
$Q_5(x) = x^5 - 4x^3 + 3x.$

The zeros of the EOBPs are all real and contained in $[-2,2]$. The Rodriguez formula for EOBPs [13] is:

$$Q_n(x) = \frac{1}{\sqrt{4-x^2}} \frac{(-1)^n (n+1)!}{(2n+1)!} \frac{d^n}{dx^n}\left[(4-x^2)^{n+\frac{1}{2}}\right], n = 0,1,2,\ldots \quad (10)$$

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The OGF for the EOBPs is still unexplored in the published literature. The orthogonality relation was developed in [13] through trigonometric substitution, whereas, a more rigorous and step-by-step approach aligned with the such proofs for the other orthogonal polynomials is presented here as a consequence of the OGF for the EOBPs.

Derivation of the OGF for EOBPs

Theorem 1. *The ordinary generating function of the EOBPs is:*

$$G(x,t) = \frac{1}{1-xt+t^2}. \tag{11}$$

Proof of Theorem 1.

The ordinary generating function of a sequence of polynomials in literature of the type: {Q n (x), n=0,1,.} can typically be written down as:

$$G(x,t) = (A+Bxt+Ct^2)^D = \sum_{n=0}^{\infty} Q_n(x)\, t^n. \tag{12}$$

In this case we have D= -1, in relation to the known families of polynomials, and only A,B and C, to be found. we expand the left side of (12) in series we have:

$$[A+(Bxt+Ct^2)]^{-1} = \left[\frac{1}{A} - \frac{(Bxt+Ct^2)}{A^2} + \frac{(Bxt+Ct^2)^2}{A^3} - \frac{(Bxt+Ct^2)^3}{A^4} + \cdots\right]. \tag{13}$$

Here, $A \neq 0$. Substituting (13) in (12), we have:

$$\frac{1}{A} - \frac{(Bxt+Ct^2)}{A^2} + \frac{(Bxt+Ct^2)^2}{A^3} - \frac{(Bxt+Ct^2)^3}{A^4} + \cdots = \sum_{n=0}^{\infty} Q_n(x)\, t^n, \tag{14}$$

or,

$$\left[\frac{1}{A} - \frac{(Bxt+Ct^2)}{A^2} + \frac{(Bxt+Ct^2)^2}{A^3} - \frac{(Bxt+Ct^2)^3}{A^4} + \cdots\right] = Q_0(x)\, t^0 + Q_1(x)\, t^1 + Q_2(x)\, t^2 + \cdots \tag{15}$$

But, we know that: $Q_0(x) = 1, Q_1(x) = x$ and $Q_2(x) = x^2 - 1$, so using these results in (15) and by comparing the coefficients of t^0, t, t^2, we have:

$$\frac{1}{A} = Q_0(x) = 1 \Rightarrow A = 1, \tag{16}$$

$$-\frac{Bx}{A^2} = Q_1(x) = x \Rightarrow B = -1, \tag{17}$$

$$\frac{B^2x^2 - C}{A^2} = x^2 - 1 \Rightarrow x^2 - C = x^2 - 1 \Rightarrow C = 1. \tag{18}$$

So, using values of A, B, C, D in (12), we get:

$$G(x,t) = \frac{1}{1-xt+t^2}. \qquad \blacksquare$$

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Alternate derivation of orthogonality relation for EOBPs

Now, we utilize the OGF proved in theorem 1, to establish alternate proof to the orthogonality relation for EOBPs in form of a consistent expression derived in [13]. The results are proved in theorem 2.

Theorem 2. *The complete orthogonality relation of EOBPs in Hilbert space* $H_Q[-2,2]$ *spanned by EOBPs is:*

$$< Q_m(x), Q_n(x) > = \int_{-2}^{2} w(x) Q_n(x)\, Q_m(x)\, dx = \pi \delta_{mn}. \tag{19}$$

Proof of Theorem 2 (alternate proof).

We know that OGF for the EOBPs is:

$$(1 - xt + t^2)^{-1} = \sum_{n=0}^{\infty} Q_n(x)\, t^n, \tag{20}$$

or,

$$(1 - xt + t^2)^{-1} = \sum_{m=0}^{\infty} Q_m(x)\, t^m. \tag{21}$$

If we multiply equations (20) and (21), then:

$$(1 - xt + t^2)^{-2} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q_m(x)\ \ Q_n(x)\, t^{m+n}. \tag{22}$$

Taking $m = n$ in (22) and weight function $w(x) = \frac{1}{2}\sqrt{4 - x^2}$ is multiplying by on both the sides, we have:

$$\frac{\sqrt{4 - x^2}}{2} \frac{1}{(1 - xt + t^2)^2} = \sum_{n=0}^{\infty} \frac{1}{2}\sqrt{4 - x^2}\, (Q_n(x))^2\, t^{2n}. \tag{23}$$

We expand $f(x,t) = \frac{1}{(1-xt+t^2)^2}$ on the left side of (23) in Maclaurin's series in t, considering x constant:

$$f(x,t) = f(x,0) + t f_t(x,0) + \frac{t^2}{2!} f_{tt}(x,0) + \frac{t^3}{3!} f_{ttt}(x,0) + \cdots, \tag{24}$$

$$f(x,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\partial^n}{\partial t^n} f(x,0). \tag{25}$$

Now we take n^{th} derivative of $f(x,t)$ with respect to t at $t = 0$, which is:

$$\frac{\partial^n}{\partial t^n} f(x,0) = \sum_{n=0}^{\infty} \frac{-2(n+2) U_n\left(\frac{x}{2}\right) + (1+n) x\, U_{n+1}\left(\frac{x}{2}\right)}{(x^2 - 1)}, \tag{26}$$

where $U_n(x)$ is Chebyshev polynomial of the 2^{nd} kind. Putting $n = 0$ in (26), we have:

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$$f(x,0) = \frac{-2(0+2)U_0\left(\frac{x}{2}\right) + (1+0)x\,U_{0+1}\left(\frac{x}{2}\right)}{(x^2-1)}. \tag{27}$$

But, $U_0\left(\frac{x}{2}\right) = 1,\ U_1\left(\frac{x}{2}\right) = 2\left(\frac{x}{2}\right), U_2\left(\frac{x}{2}\right) = 4\left(\frac{x}{2}\right)^2 - 1, U_3\left(\frac{x}{2}\right) = 8\left(\left(\frac{x}{2}\right)^3 - 4\left(\frac{x}{2}\right)\right), \ldots$

If $n = 0, f(x,0) = 1.$

If $n = 1,\ f_t(x,0) = 2x.$

If $n = 2,\ f_{tt}(x,0) = 3x^2 - 2.$

If $n = 3,\ f_{ttt}(x,0) = 2x(\,2x^2 - x)$, and so on.

To get the desired form, it is not cumbersome to put all these values in equation (25), thus we have:

$$f(x,t) = \frac{1}{(1-xt+t^2)^2} = 1 + 2xt + \ 2\,(3x^2-2\,)\frac{t^2}{2} \ + x(\,2x^2-3)\frac{t^3}{3} \ + \cdots \tag{28}$$

Now, using (28) in (23), we have:

$$\begin{aligned}
&\frac{1}{2}\sqrt{4-x^2} + \text{t x } \sqrt{4-x^2} + \ \frac{1}{2}\ \ t^2\sqrt{4-x^2}\ \ (3x^2-2\,) \\
&+ \text{t}^3\sqrt{4-\text{x}^2}\ \ (2\text{x}^2-3) + \frac{1}{2}\ \ \text{t}^4\sqrt{4-\text{x}^2}\ \ (5\text{x}^4-12\text{x}^2+3) \\
&+\ t^5 x\sqrt{4-x^2}\ \ (3x^4-10x^2+6\,) + \frac{1}{2}\ t^6\sqrt{4-x^2}\ \begin{pmatrix}7x^6-30x^4\\+30x^2-4\end{pmatrix} \\
&+ t^7 x\sqrt{4-x^2}\begin{pmatrix}4x^6-21x^4\\+30x^2-10\end{pmatrix} \\
&+ \frac{1}{2}t^8\sqrt{4-x^2}\begin{pmatrix}9x^8-56x^6\\+105x^4-60x^2\\+5\end{pmatrix} \\
&+t^9\,x\sqrt{4-x^2} + \ldots = \sum_{n=0}^{\infty}\ \frac{1}{2}\sqrt{4-x^2}\,(Q_n(x))^2\,t^{2n}\ dx.
\end{aligned} \tag{29}$$

At this point we must integrate term by term the sides between x=2 and x= -2 with respect to x in (29) and then we have:

$$I_1 = \int_{-2}^{2}\frac{1}{2}\sqrt{4-x^2}\,dx,$$

$$I_2 = \int_{-2}^{2}\frac{1}{2}\,x\sqrt{4-x^2}\,dx,$$

$$I_3 = \int_{-2}^{2}\frac{1}{2}\sqrt{4-x^2}\ (3x^2-2\,)\ dx,$$

$$I_4 = \int_{-2}^{2} x\sqrt{4-x^2}\ (2x^2-3\,)\ dx, \ldots$$

Based on even/odd functions properties and Walli formula, we can obtain:

$I_2 = I_4 = I_6 = I_8 = \cdots \ = 0$, while $I_1 = I_3 = I_5 = I_7 = I_9 = I_{11} \ldots \ = \pi.$

We use all these values in the integral result of equation (29) in order to obtain desired result.

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$$\sum_{n=0}^{\infty} t^{2n} \int_{-2}^{2} \frac{1}{2}\sqrt{4-x^2}\,(Q_n(x))^2 dx = \begin{pmatrix} \pi + \pi t^2 + \pi t^4 + \pi t^6 \\ +\pi t^8 + \pi t^{10} + \pi t^{12} + \cdots \end{pmatrix} \quad (30)$$

or,

$$\sum_{n=0}^{\infty} t^{2n} \int_{-2}^{2} \frac{1}{2}\sqrt{4-x^2}\,(Q_n(x))^2 dx = \pi \sum_{n=0}^{\infty} t^{2n}. \quad (31)$$

By comparing coefficients of t^{2n}, we obtain:

$$\int_{-2}^{2} \frac{1}{2}\sqrt{4-x^2}\,(Q_n(x))^2 dx = \pi. \quad (32)$$

Considering $m \neq n$ in (22) and multiplying the weight function both the sides, gives:

$$\frac{\sqrt{4-x^2}}{2}\frac{1}{(1-xt+t^2)^2} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^{n+m}\frac{1}{2}\sqrt{4-x^2}\,Q_n(x)Q_m(x). \quad (33)$$

Integrating both the sides with respect to x from $x = -2$ to $x = 2$, we have:

$$\int_{-2}^{2} \frac{\sqrt{4-x^2}}{2}\frac{1}{(1-xt+t^2)^2} dx = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^{n+m}\int_{-2}^{2} \frac{\sqrt{4-x^2}}{2} Q_n(x)Q_m(x)\,dx. \quad (341)$$

We already have the result of left side which is:

$$\int_{-2}^{2} \frac{\sqrt{4-x^2}}{2}\frac{1}{(1-xt+t^2)^2} dx = \pi \sum_{n=0}^{\infty} t^{2n},$$

so, we have:

$$\pi \sum_{n=0}^{\infty} t^{2n} + 0 \sum_{n=0}^{\infty}\sum_{\substack{m=0 \\ m\neq n}}^{\infty} t^n\, t^m = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^{n+m}\int_{-2}^{2} \frac{\sqrt{4-x^2}}{2}(Q_n(x)Q_m(x)\,dx. \quad (35)$$

By comparing coefficients of t^{n+m} $(n \neq m)$ in (35), we get:

$$\int_{-2}^{2} \frac{\sqrt{4-x^2}}{2}(Q_n(x)Q_m(x)dx = 0, m \neq n. \quad (36)$$

From (32) and (35), that it is beyond reasonable doubt that in the event that m=n, the inner product is π and zero otherwise. Hence,

$$< Q_m(x), Q_n(x) > = \int_{-2}^{2} \frac{1}{2}\sqrt{4-x^2}\,Q_n(x)\,Q_m(x)\,dx = \begin{cases} 0, & m \neq n \\ \pi & m = n \end{cases}. \quad \blacksquare$$

The expression proved in theorem 2 is consistent with the foundations laid in [13].

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Conclusion

In this work, two important properties, the ordinary generating function and formal orthogonality expression, of the EOBPs were investigated with proofs matching the classical approached in literature. The derived expression for the OGF for EOBPs in this study helped in alternately proving the orthogonality expression for the EOBPs. The classical expression proved in this study will help in future venues focusing more properties of the EOBPs suited to different applications.

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