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Mathematical Modeling of Lift and Drainage Problems for Thin Film Flow of a Third-Grade Fluid

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ABSTRACT

Thin film flow is a significant problem in fluid dynamics because of its wide applications in coating processes, lubrication systems, wire coating, polymer processing, food processing, chemical engineering, and related industrial operations. In many practical situations, the working fluids exhibit non-Newtonian behavior and cannot be accurately described by classical Newtonian models. Third-grade fluids are especially important because they can represent nonlinear stress–strain behavior, viscoelastic effects, and shear-dependent characteristics of complex fluids. This study develops mathematical models for lift and drainage problems involving the thin film flow of an incompressible third-grade fluid. In the lift configuration, a vertical moving belt carries a thin layer of fluid upward against gravity, whereas in the drainage configuration, the fluid flows down an inclined plane under the action of gravity. The governing equations are derived from the continuity, momentum, and energy equations under steady, laminar, and thin film assumptions. The effects of dynamic viscosity (\mu), density (\rho), film thickness (h), belt velocity (U), gravitational acceleration (g), inclination angle (\alpha), thermal conductivity (k), and third-grade material parameters (\beta_{2}) and (\beta_{3}) are considered. The main dependent quantities include velocity profile, temperature distribution, flow rate, and average velocity. The expected physical trends indicate that viscosity, density, film thickness, and inclination angle significantly influence fluid transport, while thermal conductivity controls heat distribution in the film. The proposed models provide useful insight for coating,

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lubrication, wire coating, polymer processing, food processing, and chemical engineering applications.

Keywords— Thin film flow; Third-grade fluid; Lift problem; Drainage problem; Non-Newtonian fluid; Velocity profile; Temperature distribution

Introduction

Thin film flow is a significant area of fluid dynamics because it describes the motion of a relatively thin layer of liquid over a solid surface, moving belt, inclined plane, or cylindrical body. Although the film thickness is small compared with the length of the flow domain, its behavior strongly affects the performance of many industrial and engineering systems. Thin film flow is commonly observed in coating processes, lubrication, wire coating, polymer processing, food processing, biomedical flow, chemical engineering operations, paint technology, and heat-transfer devices. In such applications, the control of film thickness, velocity distribution, flow rate, and temperature field is essential for maintaining product quality, surface uniformity, thermal stability, and mechanical efficiency.

Two important configurations in thin film fluid mechanics are lift flow over a vertical moving belt and drainage flow over an inclined plane. In the lift problem, a vertical belt moves upward through a fluid and carries a thin liquid film along its surface. Gravity acts downward and resists the upward motion of the fluid film. This situation is important in coating and withdrawal processes, where a moving surface extracts liquid from a reservoir. In the drainage problem, the fluid flows down an inclined plane under the action of gravity. This configuration appears in falling film systems, coating flows, drainage layers, and industrial transport processes. The mathematical modeling of these two cases provides useful insight into how gravity, viscosity, film thickness, belt velocity, and inclination angle affect the motion of the fluid film.

In many practical situations, the fluids involved are not Newtonian. Newtonian models assume a linear relationship between shear stress and rate of strain, but many industrial and biological fluids show nonlinear behavior. Examples include polymeric liquids, paints, oils, gels, slurries, food materials, blood, and suspensions. Bird et al. (1987) explained that non-Newtonian fluids may exhibit shear-dependent viscosity, elasticity, and normal stress effects, which cannot be properly represented by classical Newtonian models. Therefore, non-Newtonian fluid models are necessary for accurately describing the motion of complex fluids in coating, lubrication, food processing, and chemical engineering applications.

Among non-Newtonian models, third-grade fluids are especially important because they can represent nonlinear stress–strain behavior more effectively than Newtonian and second-grade fluids. Third-grade fluids contain higher-order material parameters, commonly denoted by (\beta_{2}) and (\beta_{3}), which describe nonlinear contributions to the stress tensor. Fosdick and Rajagopal (1980) provided an important thermodynamic foundation for third-grade fluids, while Siddiqui and Kaloni (1987) studied steady plane flows of third-grade fluids. These studies established the usefulness of third-grade models for analyzing complex viscoelastic fluids. Later studies by Siddiqui et al. (2006, 2008, 2013), Zahid et al. (2017), and Bhatti et al. (2019)

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further showed the importance of third-grade and related non-Newtonian models in thin film, coating, and industrial flow problems.

The mathematical modeling of third-grade thin film flow is challenging because the governing equations are nonlinear. These nonlinearities arise from the constitutive equation of the third-grade fluid, the momentum equation, and the energy equation when heat transfer effects are included. In lift and drainage problems, the velocity profile, temperature distribution, flow rate, and average velocity depend on several physical parameters, including dynamic viscosity (\mu), density (\rho), film thickness (h), belt velocity (U), gravitational acceleration (g), inclination angle (\alpha), thermal conductivity (k), and third-grade parameters (\beta_{2}) and (\beta_{3}). Hayat and Sajid (2007) noted that higher-grade fluid models often lead to nonlinear differential equations that require careful analytical or semi-analytical treatment.

Although several researchers have investigated thin film flow, coating flow, and non-Newtonian fluid motion, many studies focus either on a single configuration or on a specific solution method. Comparatively less attention has been given to presenting a unified mathematical modeling framework for both lift and drainage problems of third-grade fluid thin film flow. The thesis on which this article is based considers both vertical moving belt lift and inclined-plane drainage configurations and analyzes velocity profile, temperature distribution, flow rate, average velocity, and related physical parameters.

The objective of the present article is to develop and explain mathematical models for lift and drainage problems involving thin film flow of an incompressible third-grade fluid. The study aims to formulate the governing equations from continuity, momentum, constitutive, and energy equations and to interpret the role of major physical parameters in both configurations. The novelty of the study lies in presenting lift and drainage models within a single framework, allowing the hydrodynamic and thermal behavior of third-grade thin film flow to be compared and physically interpreted for practical engineering applications.

Literature Review

Thin Film Flow and Its Industrial Significance

Thin film flow has long been an important subject in applied mathematics and fluid mechanics because it appears in many industrial and engineering processes where a thin layer of liquid moves over a solid surface. Such flows occur in coating technology, lubrication, wire coating, polymer processing, painting, food processing, biomedical systems, chemical industries, and heat-transfer operations. In these systems, the thickness of the film, velocity distribution, surface motion, gravitational effects, and thermal behavior directly affect product quality and process efficiency. For example, in coating and wire-coating processes, a uniform film is required to achieve smooth surface finishing, while in lubrication systems the film controls frictional resistance and heat generation. Classical fluid mechanics texts and coating-flow studies, such as Bird et al. (1987), Ruschak (1985), and Kistler and Schweizer (1997), provide an important foundation for understanding thin film behavior. However, many early models were developed mainly for Newtonian fluids, which limits their application to complex industrial fluids.

Lift Flow Over Vertical Moving Belts

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Lift flow over a vertical moving belt is a significant thin film configuration in which a belt moves upward through a fluid and carries a thin liquid layer along its surface. This problem is closely related to coating, withdrawal, and industrial film-deposition processes. The upward motion of the belt attempts to lift the fluid, while gravity acts downward and tends to drain the film back. Therefore, the final velocity profile and flow rate depend on the competition between belt velocity, viscosity, density, film thickness, and gravity. Siddiqui et al. (2006) studied thin film flow of a third-grade fluid on a moving belt using homotopy perturbation methods, showing that non-Newtonian effects significantly influence the velocity field and volume flux. The strength of such studies lies in their ability to generate approximate analytical expressions, but many of them focus mainly on velocity behavior and do not fully present lift and drainage within a unified modeling framework.

Drainage Flow Over Inclined Planes

Drainage flow over an inclined plane is another important thin film problem, where gravity drives the fluid downward along a sloped surface. This configuration appears in falling film systems, drainage layers, coating flows, and chemical processing. In this case, the inclination angle plays a central role because the effective gravitational component along the plane is proportional to (g\sin\alpha). Siddiqui et al. (2008) investigated thin film flow of a third-grade fluid down an inclined plane using the Homotopy Perturbation Method and demonstrated how nonlinear material parameters affect the drainage velocity. Siddiqui et al. (2013) also examined related thin film flow problems for non-Newtonian fluids under inclined-plane and moving-belt configurations. These studies are valuable because they provide analytical or exact expressions under specific assumptions, but their main limitation is that they often treat individual configurations separately rather than developing a combined mathematical comparison of lift and drainage models.

Non-Newtonian and Third-Grade Fluid Models

Many industrial fluids do not obey Newton's law of viscosity. Polymeric liquids, paints, oils, gels, slurries, food materials, biological fluids, and suspensions may show shear-dependent viscosity, elasticity, and nonlinear stress–strain behavior. Bird et al. (1987) emphasized that such fluids require advanced constitutive relations because Newtonian models cannot adequately describe their mechanical response. Among non-Newtonian models, third-grade fluids are particularly important because they can represent nonlinear viscoelastic behavior more effectively than Newtonian and second-grade models. Fosdick and Rajagopal (1980) developed the thermodynamic foundation and stability conditions for third-grade fluids, while Siddiqui and Kaloni (1987) studied plane steady flows of third-grade fluids. These works established the mathematical and physical relevance of third-grade models. However, the inclusion of third-grade parameters (\beta_{2}) and (\beta_{3}) increases the nonlinearity of the governing equations, making exact solutions difficult.

Mathematical Modeling Approaches for Nonlinear Fluid Flow

The mathematical modeling of third-grade thin film flow usually starts from the continuity equation, momentum equation, constitutive equation, and energy equation. Under steady, laminar, incompressible, and thin film assumptions, these equations reduce to nonlinear ordinary differential equations for velocity and temperature fields.

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The thesis on which the present article is based follows this modeling approach for lift flow over a vertical belt and drainage flow over an inclined plane, with key quantities including velocity profile, temperature distribution, flow rate, average velocity, dynamic viscosity, density, film thickness, thermal conductivity, and third-grade material parameters.

Because third-grade fluid models produce nonlinear differential equations, several analytical and semi-analytical methods have been used in previous studies. These include regular perturbation methods, Homotopy Perturbation Method, Homotopy Analysis Method, Variational Iteration Method, Adomian Decomposition Method, and numerical techniques. He (2006) highlighted the usefulness of asymptotic and homotopy-based approaches for strongly nonlinear problems. Hayat and Sajid (2007) also showed that higher-grade fluid models often require approximate analytical treatment due to the complexity of their nonlinear governing equations. Zahid et al. (2017) extended third-grade fluid analysis to roll coating, while Bhatti et al. (2019) studied blade coating of a third-order fluid. These works show the wide applicability of mathematical modeling in coating and thin film processes. However, different methods may produce different levels of accuracy, convergence behavior, and computational complexity.

The reviewed literature shows that thin film flow, lift flow, drainage flow, coating flow, and third-grade fluid models have been studied extensively. However, many studies focus on either the lift problem or the drainage problem separately, or they emphasize a specific solution technique rather than the complete mathematical structure of both configurations. There remains a need for a unified modeling study that develops and interprets both lift and drainage problems for thin film flow of a third-grade fluid under one consistent framework. The present study addresses this gap by focusing on the mathematical formulation and physical interpretation of both configurations. It analyzes velocity profile, temperature distribution, flow rate, average velocity, dynamic viscosity, density, film thickness, belt velocity, inclination angle, and third-grade parameters (\beta_{2}) and (\beta_{3}). This unified approach can provide clearer insight into the behavior of non-Newtonian thin films in coating, lubrication, wire coating, polymer processing, food processing, and chemical engineering applications.

Governing Equations and Physical Assumptions

The mathematical model is developed for the thin film flow of an incompressible third-grade non-Newtonian fluid under two physical configurations: lift flow over a vertical moving belt and drainage flow over an inclined plane. The formulation follows the standard assumptions used for thin film modeling: the flow is steady, laminar, incompressible, and unidirectional; the film thickness is uniform and small compared with the characteristic length of the surface; pressure at the free surface is atmospheric; the no-slip condition is applied at the solid boundary; and the shear stress at the free surface is assumed to vanish. These assumptions are consistent with the thesis framework in which lift and drainage problems for third-grade thin film flow were modeled using continuity, momentum, constitutive, and energy equations.

Let a Cartesian coordinate system be chosen such that $x$is measured along the direction of flow and $y$is measured normal to the solid surface. For the lift problem, the solid surface is a vertical belt moving upward with constant velocity U. For the drainage

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problem, the fluid flows down an inclined plane under gravity, where $\alpha$is the inclination angle. The velocity and temperature fields are assumed as

$$\mathrm{V} = [u(y),0,0], T = T(y), \qquad (1)$$

where $u(y)$represents the velocity profile and $T(y)$or $\theta(y)$represents the temperature distribution across the film thickness.
For an incompressible fluid, the continuity equation is

$$\nabla \cdot \mathrm{V} = 0. \qquad (2)$$

Using the velocity field in Eq. (1), the continuity equation is satisfied identically because the flow is unidirectional and the velocity varies only in the normal direction. The general momentum equation for an incompressible fluid is

$$\rho \frac{D\mathrm{V}}{Dt} = \nabla \cdot \mathrm{T} + \rho \mathrm{b}, \qquad (3)$$

where $\rho$is the fluid density, Tis the Cauchy stress tensor, and bis the body force per unit mass. For steady thin film flow, the dominant body force is gravity. In the drainage problem, the gravitational component along the inclined plane is $g\sin\ \alpha$. In the lift problem, gravity acts downward and opposes the upward motion of the belt.
The Cauchy stress tensor for a third-grade fluid is written as

$$\mathrm{T} = -p\mathrm{I} + \mathrm{S}, \qquad (4)$$

where p is pressure, I is the identity tensor, and S is the extra stress tensor. The constitutive relation for a third-grade fluid is expressed as

$$\mathrm{S} = \mu\mathrm{A}_1 + \alpha_1\mathrm{A}_2 + \alpha_2\mathrm{A}_1^2 + \beta_1\mathrm{A}_3 + \beta_2(\mathrm{A}_1\mathrm{A}_2 + \mathrm{A}_2\mathrm{A}_1) + \beta_3\mathrm{tr}(\mathrm{A}_1^2)\mathrm{A}_1, \qquad (5)$$

where $\mu$is dynamic viscosity, $\alpha_1$and $\alpha_2$are second-grade material constants, and β_1, β_2, and $\beta_3$are third-grade material parameters. For the present unidirectional thin film flow, the relevant shear stress component reduces to

$$\tau_{xy} = \mu\frac{du}{dy} + 2(\beta_2 + \beta_3)\left(\frac{du}{dy}\right)^3. \qquad (6)$$

Let

$$\Gamma = \beta_2 + \beta_3. \qquad (7)$$

Then Eq. (6) becomes

$$\tau_{xy} = \mu u' + 2\Gamma(u')^3, \qquad (8)$$

where the prime denotes differentiation with respect to y. Substitution of Eq. (8) into the momentum equation gives the reduced nonlinear velocity equation

$$\frac{d\tau_{xy}}{dy} + \rho G = 0, \qquad (9)$$

or

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$$\mu u'' + 6\Gamma(u')^2 u'' + \rho G = 0. \qquad (10)$$

Here, Grepresents the effective gravitational component. For drainage flow,

$$G = g\sin\ \alpha, \qquad (11)$$

whereas for lift flow the sign of $G$is selected according to the upward belt motion and the downward action of gravity.
The energy equation for heat transfer is written as

$$\rho c_p \frac{DT}{Dt} = k\nabla^2 T + \Phi, \qquad (12)$$

where $c_p$is specific heat, $k$is thermal conductivity, and $\Phi$is the viscous dissipation term. For steady unidirectional thin film flow, Eq. (12) reduces to

$$kT'' + \Phi = 0. \qquad (13)$$

The viscous dissipation function is obtained from the product of shear stress and velocity gradient:

$$\Phi = \tau_{xy} u'. \qquad (14)$$

Using Eq. (8), this becomes

$$\Phi = \mu(u')^2 + 2\Gamma(u')^4. \qquad (15)$$

Therefore, the temperature equation is

$$kT'' + \mu(u')^2 + 2\Gamma(u')^4 = 0. \qquad (16)$$

This equation shows that temperature distribution is controlled by thermal conductivity, viscosity, third-grade parameters, and the velocity gradient.
For the lift problem, the velocity boundary conditions are

$$u(0) = U, u'(h) = 0. \qquad (17)$$

For the drainage problem, they are

$$u(0) = 0, u'(h) = 0. \qquad (18)$$

For heat transfer, suitable thermal boundary conditions may be written as

$$T(0) = T_w, T'(h) = 0, \qquad (19)$$

where $T_w$is the wall temperature and $T'(h) = 0$represents zero heat flux at the free surface.
The flow rate per unit width is defined as

$$Q = \int_0^h u(y)\, dy, \qquad (20)$$

and the average film velocity is

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$$\bar{u} = \frac{Q}{h} = \frac{1}{h}\int_0^h u(y)\, dy. \qquad (21)$$

Thus, μ, ρ, h, β_2, β_3, k, T, U, g, α, Q, and $\bar{u}$form the main physical quantities of the model. Equations (10) and (16), together with the stated boundary conditions, provide the mathematical basis for analyzing lift and drainage thin film flow of a third-grade fluid.

Mathematical Modeling of the Lift Problem

The lift problem considers a vertical moving belt passing through an incompressible third-grade fluid and carrying a thin layer of fluid upward along its surface. During this motion, the belt moves with a constant velocity U, while gravity acts downward and opposes the upward transport of the liquid film. This configuration is important in coating, withdrawal, lubrication, and polymer-processing operations where a moving surface draws a thin layer of non-Newtonian fluid from a reservoir. The thesis model also treats this configuration as one of the main cases for third-grade thin film flow analysis.

Let a Cartesian coordinate system be chosen such that $x$is measured normal to the belt surface and $y$is measured vertically upward along the direction of belt motion. The fluid film has uniform thickness h, where $x = 0$ represents the belt surface and $x = h$represents the free surface of the film. Since the flow is assumed steady, laminar, incompressible, and unidirectional, the velocity field and temperature field may be written as

$$\mathrm{V} = [0, v(x), 0], T = T(x), \qquad (22)$$

where $v(x)$ denotes the upward velocity profile and $T(x)$ denotes the temperature distribution across the film thickness.

For an incompressible fluid, the continuity equation is

$$\nabla \cdot \mathrm{V} = 0. \qquad (23)$$

Using the velocity field in Eq. (22), the continuity equation is satisfied identically because the only non-zero velocity component depends only on the transverse coordinate x. Hence, the flow model is reduced to the momentum balance in the vertical direction.

The momentum equation in the flow direction may be written as

$$\frac{d\tau_{xy}}{dx} - \rho g = 0, \qquad (24)$$

where $\tau_{xy}$is the shear stress, $\rho$is the fluid density, and $g$is gravitational acceleration. The negative sign represents the fact that gravity acts downward, opposite to the upward motion of the belt.

For a third-grade fluid, the shear stress relation for the present unidirectional flow is expressed as

$$\tau_{xy} = \mu\frac{dv}{dx} + 2(\beta_2 + \beta_3)\left(\frac{dv}{dx}\right)^3, \qquad (25)$$

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where $\mu$is the dynamic viscosity and $\beta_2$and $\beta_3$are third-grade material parameters. Introducing

$$\Gamma = \beta_2 + \beta_3, \qquad (26)$$

Eq. (25) becomes

$$\tau_{xy} = \mu v' + 2\Gamma(v')^3, \qquad (27)$$

where the prime denotes differentiation with respect to x. Substituting Eq. (27) into Eq. (24) gives the reduced nonlinear ordinary differential equation for the lift velocity:

$$\mu v'' + 6\Gamma(v')^2 v'' - \rho g = 0. \qquad (28)$$

Equation (28) represents the balance between viscous resistance, nonlinear third-grade fluid effects, and gravitational opposition. The nonlinear term $6\Gamma(v')^2 v''$is due to the third-grade nature of the fluid.

The velocity boundary conditions are based on the no-slip condition at the belt and the shear-free condition at the free surface. At the belt surface, the fluid adheres to the moving belt, so

$$v(0) = U. \qquad (29)$$

At the free surface, the shear stress vanishes:

$$\tau_{xy}(h) = 0. \qquad (30)$$

Using Eq. (27), the shear-free condition may be written as

$$v'(h) = 0, \qquad (31)$$

provided the shear relation remains regular.

The temperature distribution is governed by the energy equation with viscous dissipation. For steady unidirectional flow, the reduced energy equation is

$$kT'' + \Phi = 0, \qquad (32)$$

where $k$is the thermal conductivity and $\Phi$is the viscous dissipation term. Since

$$\Phi = \tau_{xy} v', \qquad (33)$$

substitution of Eq. (27) gives

$$\Phi = \mu(v')^2 + 2\Gamma(v')^4. \qquad (34)$$

Thus, the temperature equation becomes

$$kT'' + \mu(v')^2 + 2\Gamma(v')^4 = 0. \qquad (35)$$

Suitable thermal boundary conditions may be written as

$$T(0) = T_w, T'(h) = 0, \qquad (36)$$

where $T_w$is the belt temperature and $T'(h) = 0$represents zero heat flux at the free

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surface.
The flow rate per unit width is defined by

$$Q = \int_0^h v(x)\, dx, \qquad (37)$$

and the average film velocity is

$$\bar{v} = \frac{Q}{h} = \frac{1}{h}\int_0^h v(x)\, dx. \qquad (38)$$

In the lift problem, μcontrols viscous resistance and film attachment to the moving belt, ρdetermines the gravitational load opposing upward motion, $h$affects the amount of fluid carried, and Uprovides the driving motion. The parameters $\beta_2$and $\beta_3$describe nonlinear non-Newtonian behavior, while $k$controls heat diffusion within the film. Thus, the lift model connects the hydrodynamic and thermal behavior of a third-grade fluid film carried upward by a vertical moving belt.

Mathematical Modeling of the Drainage Problem

The drainage problem describes the steady laminar motion of an incompressible third-grade fluid flowing as a thin film down an inclined plane under the action of gravity. This configuration is important in coating layers, falling film systems, drainage processes, lubrication, polymer processing, and chemical engineering applications. In this model, gravity acts as the driving force, while viscous resistance and non-Newtonian material effects control the rate at which the fluid drains. The thesis framework also considers this drainage configuration as one of the central thin film flow problems for a third-grade fluid.
Let a Cartesian coordinate system be selected such that $x$is measured along the inclined plane in the direction of flow, and $y$is measured normal to the plane. The inclination angle of the plane is denoted by α, so the gravitational component along the plane is $g\sin\,\alpha$. The fluid film has uniform thickness h, where $y = 0$represents the solid inclined surface and $y = h$represents the free surface of the film.
The flow is assumed to be steady, laminar, incompressible, and unidirectional. Therefore, the velocity and temperature fields are written as

$$\mathrm{V} = [u(y),0,0], T = T(y), \qquad (39)$$

where $u(y)$ is the velocity component along the inclined plane and $T(y)$ is the temperature distribution across the film thickness.
For an incompressible fluid, the continuity equation is

$$\nabla \cdot \mathrm{V} = 0. \qquad (40)$$

Using the velocity field in Eq. (39), the continuity equation is satisfied automatically because the velocity has only one non-zero component and depends only on the transverse coordinate y. Thus, the governing model is reduced to the momentum balance in the x-direction.
The momentum equation along the inclined plane is expressed as

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$$\frac{d\tau_{xy}}{dy} + \rho g \sin\ \alpha = 0, \qquad (41)$$

where $\tau_{xy}$is the shear stress, $\rho$is the fluid density, $g$is gravitational acceleration, and $\rho g \sin\ \alpha$represents the gravitational force per unit volume acting along the inclined plane. Since the pressure at the free surface is assumed atmospheric and no external pressure gradient is applied, the motion is mainly gravity-driven.

For a third-grade fluid, the shear stress relation for the present unidirectional flow is

$$\tau_{xy} = \mu \frac{du}{dy} + 2(\beta_2 + \beta_3)\left(\frac{du}{dy}\right)^3, \qquad (42)$$

where μ is dynamic viscosity, and β_2 and β_3 are third-grade material parameters. Introducing

$$\Gamma = \beta_2 + \beta_3, \qquad (43)$$

Eq. (42) becomes

$$\tau_{xy} = \mu u' + 2\Gamma(u')^3, \qquad (44)$$

where the prime denotes differentiation with respect to y. Substituting Eq. (44) into Eq. (41), the reduced nonlinear ordinary differential equation for the drainage velocity is obtained as

$$\mu u'' + 6\Gamma(u')^2 u'' + \rho g \sin\ \alpha = 0. \qquad (45)$$

Equation (45) represents the balance between gravitational driving force, viscous resistance, and nonlinear third-grade fluid effects. The nonlinear term $6\Gamma(u')^2u''$arises due to the third-grade nature of the fluid.

The velocity boundary conditions are based on the no-slip condition at the inclined plane and the shear-free condition at the free surface. At the solid surface,

$$u(0) = 0, \qquad (46)$$

which means the fluid adheres to the stationary inclined plane. At the free surface, the shear stress is assumed to vanish:

$$\tau_{xy}(h) = 0. \qquad (47)$$

Using Eq. (44), this condition may be written as

$$u'(h) = 0. \qquad (48)$$

The temperature field is governed by the energy equation with viscous dissipation. For steady unidirectional flow, the reduced energy equation is

$$kT'' + \Phi = 0, \qquad (49)$$

where k is thermal conductivity and Φ is the viscous dissipation function. The dissipation term is obtained from

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$$\Phi = \tau_{xy} u', \qquad (50)$$

which gives

$$\Phi = \mu(u')^2 + 2\Gamma(u')^4. \qquad (51)$$

Thus, the temperature equation becomes

$$kT'' + \mu(u')^2 + 2\Gamma(u')^4 = 0. \qquad (52)$$

Suitable thermal boundary conditions may be written as

$$T(0) = T_w, T'(h) = 0, \qquad (53)$$

where T_w is the wall temperature and $T'(h) = 0$represents zero heat flux at the free surface.

The flow rate per unit width is defined as

$$Q = \int_0^h u(y)\, dy, \qquad (54)$$

and the average velocity of the film is

$$\bar{u} = \frac{Q}{h} = \frac{1}{h}\int_0^h u(y)\, dy. \qquad (55)$$

In the drainage problem, μresists downward motion, while ρ, g, and αstrengthen the gravitational driving force. The film thickness $h$controls the total fluid volume available for transport. The parameters β_2 and β_3 describe nonlinear non-Newtonian effects, while $k$controls heat diffusion within the film. Thus, the drainage model provides a complete mathematical framework for studying velocity, temperature, flow rate, and average velocity of a third-grade fluid flowing down an inclined plane.

Results and Physical Discussion

The mathematical models for lift and drainage thin film flow of a third-grade fluid provide useful insight into how hydrodynamic and thermal fields respond to changes in the governing physical parameters. The two configurations are physically opposite in nature. In the lift problem, the vertical belt moves upward with velocity $U$and attempts to carry the fluid film against gravity. In the drainage problem, gravity drives the fluid downward along an inclined plane. Therefore, the same parameter may influence the two cases differently. The thesis framework also treats lift and drainage as the two main thin film flow problems and analyzes velocity profile, temperature distribution, flow rate, average velocity, and net upward flow.

Dynamic viscosity μhas a direct effect on the velocity field because it measures the internal resistance of the fluid. In the lift configuration, the model indicates that higher viscosity may support film attachment to the moving belt and help resist downward drainage. A more viscous fluid is dragged more effectively by the belt surface, particularly near the wall where the no-slip condition applies. However, in the drainage configuration, the trend is opposite. Since the fluid is driven downward by gravity, higher viscosity resists the motion and reduces the drainage velocity. Thus, viscosity

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assists the retention of fluid in the lift problem but slows the gravitational motion in the drainage problem. This contrast is consistent with the thesis discussion, where drainage velocity decreases with increasing dynamic viscosity, while drainage increases with density, film thickness, and inclination angle.

The density ρalso affects the two configurations differently. In the lift problem, gravity acts against the upward motion of the belt, so increasing density increases the downward gravitational load on the film. As a result, the belt must overcome a stronger opposing force to carry the fluid upward. Therefore, the model suggests that higher density may reduce net upward transport if the belt velocity is not sufficiently large. In the drainage problem, however, density strengthens the gravitational driving force $\rho g \sin\ \alpha$. Hence, increasing density tends to increase the drainage velocity, flow rate, and average velocity. This behavior shows that density acts as a resisting factor in lift flow but as a driving factor in drainage flow.

Film thickness $h$plays an important role in determining the quantity of fluid transported. Since the flow rate is defined as

$$Q = \int_0^h u(y)\, dy,$$

a larger film thickness generally increases the volume of fluid available for motion. In drainage flow, increasing $h$may increase the total flow rate because a thicker film contains more fluid and allows a larger velocity distribution across the film depth. In lift flow, a thicker film can increase the amount of fluid carried by the belt, but it may also increase the weight of the film and therefore the resistance against upward motion. The average velocity,

$$\bar{u} = \frac{Q}{h},$$

should be interpreted carefully because it represents the mean velocity across the film thickness rather than the total volume transported. Thus, a thicker film may increase Q, but the effect on $\bar{u}$depends on how the velocity profile changes with h.

The belt velocity $U$is a key parameter in the lift problem. The no-slip condition requires that the fluid velocity at the belt surface equals the belt velocity. Therefore, an increase in $U$is expected to increase the velocity near the belt and strengthen upward film transport. Physically, a faster belt supplies more mechanical driving force to the fluid layer. However, the upward transport is still opposed by gravity and modified by viscosity, density, and third-grade material effects. If $U$is sufficiently large, the belt can carry a thicker or denser film upward. If $U$is small, the gravitational drainage tendency may dominate, reducing net upward flow.

In the drainage problem, the inclination angle $\alpha$is one of the most important parameters. The gravitational component along the plane is $g \sin\ \alpha$, so an increase in αincreases the driving force along the inclined surface. The model indicates that larger inclination angles produce higher drainage velocity, larger flow rate, and greater average velocity. At small inclination angles, the gravitational component is weak and the film drains slowly. At larger angles, the film moves more rapidly. The thesis also reports that the

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third-grade fluid drains quickly with increasing uniform thickness, density, and inclined angle, while it drains slowly with increasing dynamic viscosity and material parameters.

The third-grade parameters β_2 and β_3 represent nonlinear non-Newtonian effects in the fluid. Their combined contribution usually appears through $\beta_2 + \beta_3$. When this combined effect increases, the fluid behavior deviates more strongly from the Newtonian case. The nonlinear stress terms modify the shear stress distribution and therefore influence both velocity and temperature profiles. If $\beta_2 + \beta_3 = 0$, the nonlinear third-grade contribution vanishes and the model reduces toward the Newtonian limiting case. This limiting condition is important because it confirms the physical consistency of the model.

Thermal conductivity k controls the diffusion of heat in the film. The temperature equation includes viscous dissipation, which is generated by internal friction and velocity gradients. When k is large, heat diffuses more effectively and the temperature rise is moderated. When k is small, heat remains concentrated inside the film, producing a higher temperature distribution. The thesis results similarly indicate that temperature increases when thermal conductivity decreases and when parameters such as viscosity, density, film thickness, and inclination angle increase.

The velocity profile describes how the fluid speed changes from the solid surface to the free surface. In lift flow, the maximum influence of belt motion occurs near the wall because of the no-slip condition. In drainage flow, gravity drives the fluid downward, while the no-slip condition fixes the velocity at the inclined plane and the shear-free condition controls the free surface behavior. The temperature profile is closely linked with the velocity gradient. Larger velocity gradients produce stronger viscous dissipation and therefore higher thermal effects. Thus, the hydrodynamic and thermal fields are coupled.

Table 1

Effect of μ on lift and drainage velocity

μ	Lift velocity trend	Drainage velocity trend	Physical interpretation
μ_1	To be computed	To be computed	Lower viscous resistance
μ_2	To be computed	To be computed	Moderate viscous effect
μ_3	To be computed	To be computed	Higher resistance to drainage

Table 2

Effect of h on flow rate and average velocity

h	Flow rate Q	Average velocity $\bar{u}$	Physical interpretation
h_1	To be computed	To be computed	Thin film transport
h_2	To be computed	To be computed	Moderate film thickness
h_3	To be computed	To be computed	Larger fluid volume

Table 3

Effect of α on drainage velocity and flow rate

α	Drainage velocity	Flow rate Q	Average velocity $\bar{u}$
α_1	To be computed	To be computed	To be computed

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α	Drainage velocity	Flow rate Q	Average velocity $\bar{u}$
α_2	To be computed	To be computed	To be computed
α_3	To be computed	To be computed	To be computed

Overall, the physical interpretation shows that lift flow is mainly controlled by belt velocity, viscosity, density, and gravity, while drainage flow is mainly controlled by density, inclination angle, viscosity, and film thickness. These models are useful for coating, lubrication, wire coating, food processing, polymer processing, and chemical engineering systems where control of film thickness, velocity, heat transfer, and flow rate is essential.

Conclusion

The present study developed mathematical models for lift and drainage problems involving thin film flow of an incompressible third-grade fluid. The modeling framework was based on the fundamental equations of fluid mechanics, including the continuity equation, momentum equation, constitutive relation for a third-grade fluid, and energy equation for heat transfer. Through these governing equations, the study provided a systematic formulation for analyzing the hydrodynamic and thermal behavior of non-Newtonian thin film flow under two important physical configurations. The thesis framework also considers both vertical belt lift and inclined-plane drainage as central problems for third-grade thin film flow analysis.

In the lift problem, a vertical moving belt carries a thin layer of fluid upward, while gravity acts downward and opposes the motion of the film. In the drainage problem, the fluid flows down an inclined plane under the action of gravity. For both cases, the velocity profile, temperature distribution, flow rate, and average velocity were considered as the main dependent quantities. These quantities are important for understanding fluid transport, heat generation, and film stability in practical applications.

The analysis shows that dynamic viscosity, density, film thickness, belt velocity, inclination angle, thermal conductivity, and third-grade material parameters (\beta_{2}) and (\beta_{3}) significantly influence the flow and thermal behavior of the fluid film. Dynamic viscosity controls internal resistance, density modifies gravitational effects, film thickness affects total fluid transport, belt velocity drives the lift process, and inclination angle controls the drainage rate. Thermal conductivity regulates heat diffusion, while (\beta_{2}) and (\beta_{3}) represent the nonlinear non-Newtonian behavior of the third-grade fluid.

The developed models are useful for coating processes, lubrication systems, wire coating, polymer processing, food processing, and chemical engineering operations. Future work may extend the present formulation through numerical validation, experimental verification, magnetohydrodynamic effects, nanofluid models, slip boundary conditions, variable viscosity, surface tension effects, and fractional-order fluid models.

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