Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 4 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189

http://amresearchreview.com/index.php/Journal/about

# A Highly Efficient Third-Order Integrator for Initial Value Problems

**Muhammad Daud Kandhro***
Institute of Mathematics and Computer Science, University of Sindh, Jamshoro.
School Education and Literacy Department (SELD), Govt. of Sindh.
Email of the corresponding author*: daud.kandhro@gmail.com

**Irfan Ali Lashari**
Aror University of Art, Architecture, Design and Heritage, Sukkur, Sindh, Pakistan.
Email: irfanali.faculty@aror.edu.pk

**Sobia Vighio**
Government Girls Degree College Qasimabad Hyderabad, College Education Department Government of Sindh, Pakistan
Email: sobianoor022@gmail.com

**Asad Ali Khore**
Institute of Mathematics & Computer Science, University of Sindh, Jamshoro, Pakistan.
**Email:** asadcs102@gmail.com

## ABSTRACT

In this article, new third order well-organized numerical technique for IVP of ODE's including partial derivative has constructed and enhanced its competency regarding truncation error. Taylor's series is used to derive and develop new family of numerical iterative integrative. This new iterative integrator is used to evaluate numerically ODE's with initial condition. Numerical examples are tested to check performance of each methods. Results in term of absolute error are available in each table present in this paper. Last Absolute Error, Maximum Absolute Error and CPU Time has calculated which shows and tells proposed integrator is better in all other available methods in literature. Convergence of proposed iterative integrator method is diagnosed by testing several problems and its consequence verified with available methods. Numerical results are given for proposed METHOD to illustrate the accuracy and efficiency. The comparisons of the proposed and existing schemes having same order of local accuracy are discussed. Based on the results, the proposed iterative integrator method gives the better result than few same order methods. MATLAB 2023a is preferred for numerically and graphically analysis of methods.

**Key Words:** Numerical Methods, Third order, Local Truncation Error, Convergence

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 4 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189

http://amresearchreview.com/index.php/Journal/about

## INTRODUCTION

Several physical problems in natural-science, mathematical physics and in civil-engineering are formulated by differential equations to name a few (Dennis G. Zill, Warren S and Peter V. O'Neil). Mathematical models have constructed to identify the nature of problems; these are mostly in differential form. So for, sometimes huge number of complexity arises to handle and solve models. Many of them need a solution in a numerical way. Researchers day by day are trying to innovate many models and methods to obtain better consequences. But still huge number of work is needed to understand proper behaves of models. Models have distinct nature, but here we only talk about those models which have derivative. To get solution of these models we require approximate methods. Researchers and scholars are out on the field of innovation to hone their skills. But they haven't been able succeed properly yet. Many of them have also done a great job to construct and modified new methods. Methods have different capabilities, sometimes they are numerically and sometimes they are analytical. But every model doesn't give analytical result so in such cases we preferred numerical methods. But drawback of numerical method is that they don't give reliable result because always left some numbers called error. Few families such as RK and Euler's has sub-families numerically. They have different converging order. Here our main focus of third converging order method. So work on third converging order, few methods tested in sense of better performance and reduce length of error. Different methods have nature separately. Some are explicit and other are implicit. Implicit is more time taken instead of explicit but it approaches quickly to the true result. Researchers day by day showing their better interest to enhance convergence, order and stability. Many equations named differential don't reach to the solution such as particular and analytical. To deal with such difficulty new innovations take place in numerical analysis. Best advantage of numerical scheme is that these have better performance than analytical.

## MATERIALS AND METHODS

A Taylor series or Taylor sum is a representation of mathematical function form of series containing terms calculated by utilizing values of the function's derivation at a specific point, we have

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 4 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189
http://amresearchreview.com/index.php/Journal/about

$$G(x_n, y_n) = y(x) + hf + h^2\left[\frac{1}{2}f_x + \frac{1}{2}f_y f\right] + h^3\left[\begin{array}{l}\frac{1}{6}f_{xx} + \frac{1}{6}f_x f_y + \\ \frac{1}{3}\left(f_{xy} + \frac{1}{2}f_{yy}f + \frac{1}{2}f_y^2\right)f\end{array}\right]$$

$$+h^4\left[\begin{array}{l}\frac{1}{24}f_{xxx} + \frac{1}{24}\left(3f_{xxy} + 3f_{xyy}f + 5f_y f_{xy}\right)f \\ +\frac{1}{8}f_x f_{xy} + \frac{1}{24}f_{yyy}f^3 + + \frac{1}{24}\left(f_x f_y^2 + f_y f_{xx}\right) \\ \frac{1}{24}\left(3f_y f^2 f_{yy} + 3f_x f f_{yy} + f_y^3\right)f\end{array}\right] + O(h^5) \qquad (1)$$

This (1) is Taylor's Series which is one of the most useful and authentication tool for many purposes like as new methods, order of convergence etc. can be derived. Numerical methods have designed to attain the estimated consequences of IVP's for differential equation in nature ordinary. We can consider general function form of problems:

$$\frac{dy}{dx} = f(x_n, y_n), \quad y(x_0) = y_0 \qquad (2)$$

$$y_{n+1} = y_n + \int_{x_0}^{x_0+h} f(x_n,\, y_n)\, dx$$

In discrete manner

$$y_{n+1} = y_n + h\sum_{i=1}^{s}\rho_i w_i$$

If $i = 1,2,3$

$$y_{n+1} = y_n + h(\rho_1 w_1 + \rho_2 w_2 + \rho_3 w_3) \qquad (3)$$

Where $w_1, w_2$ and $w_3$ are the slopes, determined by

$$\begin{aligned} w_1 &= f(x_n, y_n) \\ w_2 &= f(x_n + \alpha_2 h, y_n + h(\beta_{21} w_1) + h^2(\delta_{21} w_1) f_y) \\ w_3 &= f(x_n + \alpha_3 h, y_n + h(\beta_{31} w_1 + \beta_{32} w_2) + h^2(\delta_{31} w_1) f_y) \end{aligned} \qquad (4)$$

Slopes $w_2$ and $w_3$ are expended thru using Taylor's Series then surrogate $w_1, w_2$ and $w_3$ into $(3)$. Finally, coefficients of are equating with $(1)$ in power of $h$ up to $h^3$ for attaining system.

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 4 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189
http://amresearchreview.com/index.php/Journal/about

$$\rho_1+\rho_2+\rho_3=1 \qquad \alpha_2\rho_3\beta_{32}=\frac{1}{6} \qquad \alpha_2\rho_2+\alpha_3\rho_3=\frac{1}{2}$$
$$\frac{1}{2}\left(\alpha_2^2\rho_2+\alpha_3^2\rho_3\right)=\frac{1}{6} \qquad \rho_2\beta_{21}+\rho_3\beta_{31}+\rho_3\beta_{32}=\frac{1}{2} \qquad (5)$$
$$\alpha_2\rho_2\beta_{21}+\alpha_3\rho_3\beta_{31}+\alpha_3\rho_3\beta_{32}=\frac{1}{3} \qquad \rho_2\delta_{21}+\rho_3\delta_{31}+\rho_3\beta_{21}\beta_{32}=\frac{1}{6}$$
$$\frac{1}{2}\left(\rho_2\beta_{21}^2+\rho_3\beta_{31}^2+\rho_3\beta_{32}^2\right)+\rho_3\beta_{31}\beta_{32}=\frac{1}{6}$$

Above system is nonlinear which has 8 equations and 10 variable quantities. Now we investigate this system for solution. So it has multiple solution because of free variables. One of the accurate and reliable solution we have

$$\begin{aligned} w_1 &= f\left(x_n, y_n\right) \\ w_2 &= f\left(x_n+\frac{2h}{3}, y_n+hw_1\left(\frac{2}{3}-3hf_y\right)\right) \\ w_3 &= f\left(x_n+\frac{2h}{3}, y_n-\frac{5}{6}hw_1+\frac{3}{2}hw_2+\frac{21}{2}h^2w_1f_y\right) \\ y_{n+1} &= y_n+\frac{h}{12}\left(3w_1+7w_2+2w_3\right) \end{aligned} \qquad (6)$$

This is new integrator method having three function evaluations per time step.

## ERROR ANALYSIS

It is main purpose to solve ordinary differential equation numerically to attain results which are as close as possible to the exact solution. Two sources of error which affect the accuracy of numerical method named as truncation (TR) and round-off (RO). TR arises because mathematical procedures are approximated (e.g., by deserting higher-order terms), while round-off error takes the place when computers can only stock numbers with limited precision. An accuracy can totally rely on blunder what size of step size taken. Some schemes are accepted for convergence just we note how quickly reach to true point. The convergence of IVP is calculated by $e_n = |y(x_m) - y_m| < \delta$ where approximate consequence subtract from exact and A practical error bound '$\delta$' is a predefined tolerance level that limits how large the error in a numerical solution is allowed to be during computation. Therefore, $\delta$ relies nature and consequences of problems, it differs from $10^{-3}$. The errors for numerical schemes; $Error = |y(x_m) - y_m|$

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 4 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189

http://amresearchreview.com/index.php/Journal/about

## ORDER OF CONVERGENCE

New accelerated proposed scheme is compared numerically and graphically with other accessible techniques taken from literature with accuracy of 3rd order. So, New accelerated proposed scheme is behaving numerically better than other third order existing schemes in literature. So from the numerical result we observe new accelerated proposed scheme has third order of convergent.

## NUMERICAL EXPERIMENTS

The performance of new accelerated technique has shown in opposite of few already existing techniques present in open literature, we take few Cauchy problems and implement numerical scheme including new proposed scheme on that problems to diagnose which scheme shows better performance in all criteria of better accuracy. Different errors like as error absolute including maximum and last are acquired at the latest junction point of the interval and CPU (process time) has calculated for claiming new accelerated proposed scheme is better at all. We tested few IVP of ODE's to check the rendition of the new accelerated method in contrast other. Two renowned techniques named in abbreviation as RK3MH [7] and RK3MA [5] are nominated for checking the performance in against of accelerated proposed method. MATLAB R2023a is used to analyze result and visualize its graph.

**Table 1: Shows Numerically Illustration of Problem 1**

**Problem 1**

$\frac{dy}{dx} = xy^3 - y, \quad y(0) = 1, \qquad Exact = \frac{2}{\sqrt{2 + 4x + 2e^{2x}}}$

| Step-Size/ Method | | RK3MH | RK3MA | Proposed |
|---|---|---|---|---|
| 0.1 | $Error_{(Max)}$ | 2.2406e-04 | 0.0052 | 9.9307e-05 |
| | $Error_{(Last)}$ | 2.1363e-04 | 0.0052 | 9.9307e-05 |
| | $Time_{(PT)}$ | 0e+0 | 0e+0 | 0e+0 |
| 0.05 | $Error_{(Max)}$ | 5.1698e-05 | 0.0027 | 5.3473e-06 |
| | $Error_{(Last)}$ | 4.9489e-05 | 0.0027 | 1.3432e-06 |
| | $Time_{(PT)}$ | 0e+0 | 0e+0 | 0e+0 |
| 0.025 | $Error_{(Max)}$ | 1.2497e-05 | 0.0014 | 9.7900e-07 |
| | $Error_{(Last)}$ | 1.1988e-05 | 0.0014 | 7.7144e-07 |
| | $Time_{(PT)}$ | 0e+0 | 0e+0 | 0e+0 |
| 0.0125 | $Error_{(Max)}$ | 3.0773e-06 | 6.9019e-04 | 1.4366e-07 |
| | $Error_{(Last)}$ | 2.9553e-06 | 6.9019e-04 | 1.2837e-07 |
| | $Time_{(PT)}$ | 0e+0 | 0e+0 | 0e+0 |

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 4 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189

http://amresearchreview.com/index.php/Journal/about

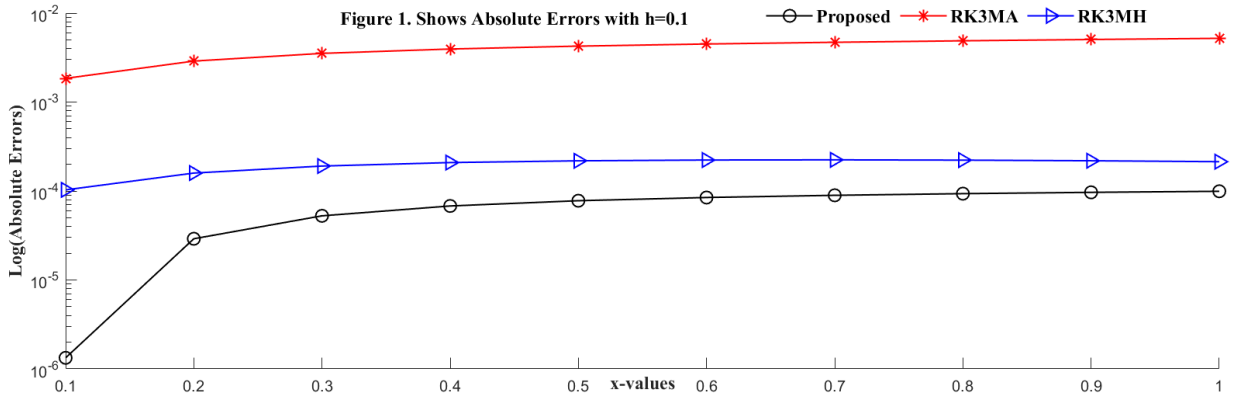



**Table 2: Shows Numerically Illustration of Problem 2**

| **Problem 2** $\frac{dy}{dx} = 3x + \frac{y}{2}, \quad y(0) = 1 \quad Exact = 13e^{\frac{x}{2}} - 6x - 12$ | | | | |
|---|---|---|---|---|
| **Step-Size/Method** | | **RK3MH** | **RK3MA** | **Proposed** |
| 0.1 | $Error_{(Max)}$ | 0.0051 | 0.0336 | 2.9760e-04 |
| | $Error_{(Last)}$ | 0.0051 | 0.0336 | 2.9760e-04 |
| | $Time_{(PT)}$ | 0e+0 | 0e+0 | 0e+0 |
| 0.05 | $Error_{(Max)}$ | 0.0013 | 0.0175 | 3.9233e-05 |
| | $Error_{(Last)}$ | 0.0013 | 0.0175 | 3.9233e-05 |
| | $Time_{(PT)}$ | 0e+0 | 0e+0 | 0e+0 |
| 0.025 | $Error_{(Max)}$ | 3.2374e-04 | 0.0090 | 5.0345e-06 |
| | $Error_{(Last)}$ | 3.2374e-04 | 0.0090 | 5.0345e-06 |
| | $Time_{(PT)}$ | 0e+0 | 0e+0 | 0e+0 |
| 0.0125 | $Error_{(Max)}$ | 8.1035e-05 | 0.0045 | 6.3757e-07 |
| | $Error_{(Last)}$ | 8.1035e-05 | 0.0045 | 6.3757e-07 |
| | $Time_{(PT)}$ | 0e+0 | 0e+0 | 0e+0 |

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 4 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189

http://amresearchreview.com/index.php/Journal/about

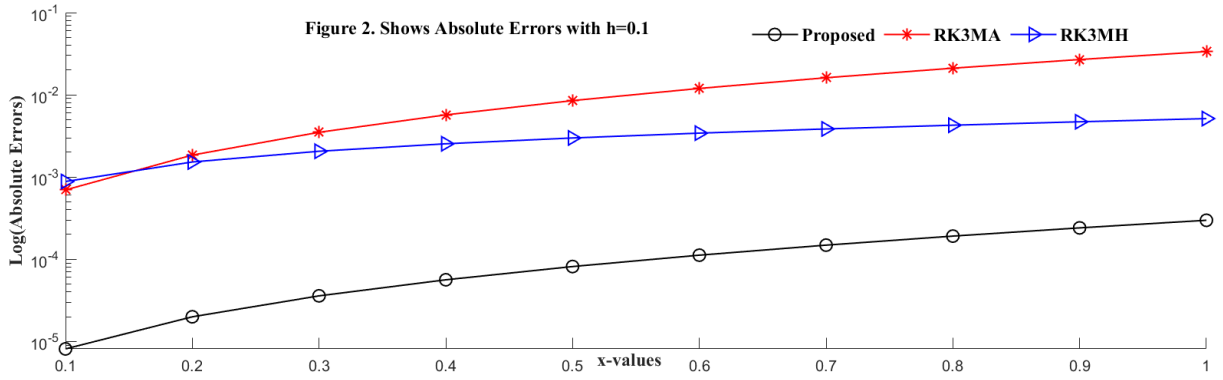



Figure 2. Shows Absolute Errors with h=0.1

**Table 3: Shows Numerically Illustration of Problem 3**

**Problem 3.**

$\frac{dy}{dx} = \frac{\sin x}{y}, \quad y(0) = 1, \quad Exact = \sqrt{3 - 2\cos x}$

| Step-Size/Methods | | RK3MH | RK3MA | Proposed |
|---|---|---|---|---|
| 0.1 | $Error_{(Max)}$ | 0.0023 | 0.0033 | 3.7702e-05 |
| | $Error_{(Last)}$ | 0.0020 | 0.0033 | 3.7702e-05 |
| | $Time_{(PT)}$ | 0e+0 | 0e+0 | 0e+0 |
| 0.05 | $Error_{(Max)}$ | 6.5681e-04 | 0.0016 | 3.7484e-06 |
| | $Error_{(Last)}$ | 5.5494e-04 | 0.0016 | 3.7484e-06 |
| | $Time_{(PT)}$ | 0e+0 | 0e+0 | 0e+0 |
| 0.025 | $Error_{(Max)}$ | 1.8632e-04 | 8.2142e-04 | 4.0519e-07 |
| | $Error_{(Last)}$ | 1.5560e-04 | 8.2142e-04 | 4.0519e-07 |
| | $Time_{(PT)}$ | 0e+0 | 0e+0 | 0e+0 |
| 0.0125 | $Error_{(Max)}$ | 5.2173e-05 | 4.1013e-04 | 4.6585e-08 |
| | $Error_{(Last)}$ | 4.3185e-05 | 4.1013e-04 | 4.6585e-08 |
| | $Time_{(PT)}$ | 0e+0 | 0e+0 | 0e+0 |

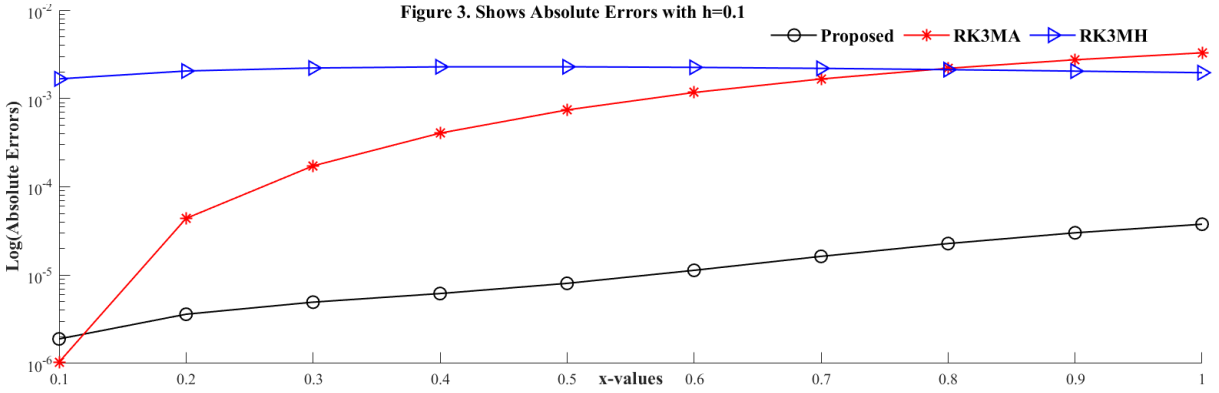


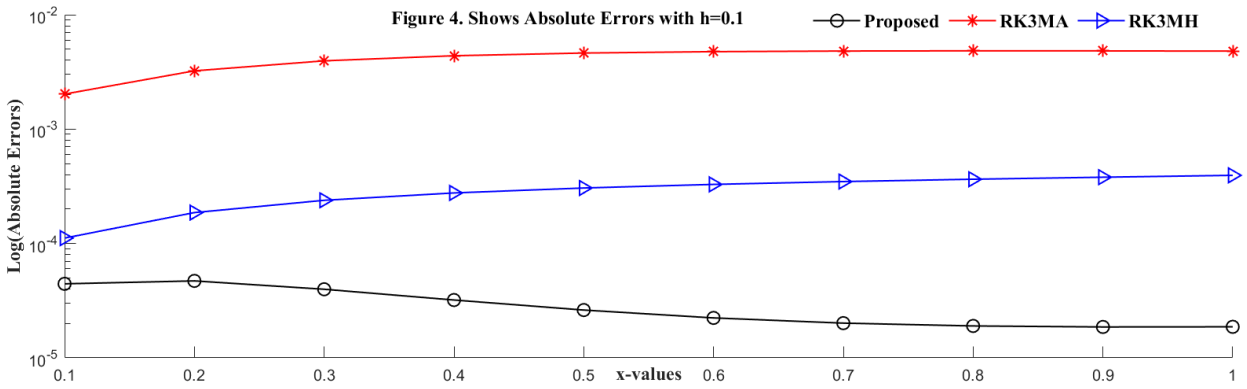

Figure 3. Shows Absolute Errors with h=0.1

Annual Methodological Archive Research Review
http://amresearchreview.com/index.php/Journal/about
Volume 4, Issue 4 (2026)

Online ISSN 3007-3197
Print ISSN 3007-3189
http://amresearchreview.com/index.php/Journal/about

**Table 4: Shows Numerically Illustration of Problem 3**

| **Problem 4.** $\frac{dy}{dx} = xy^2 - y, \quad y(0) = 1, \quad Exact = \frac{1}{1+x}$ | | | | |
|---|---|---|---|---|
| **Step-Size/Methods** | | **RK3MH** | **RK3MA** | **Proposed** |
| 0.1 | $Error_{(Max)}$ | 3.9431e-04 | 0.0048 | 4.7101e-05 |
| | $Error_{(Last)}$ | 3.9431e-04 | 0.0048 | 1.8691e-05 |
| | $Time_{(PT)}$ | 0e+0 | 0e+0 | 0e+0 |
| 0.05 | $Error_{(Max)}$ | 9.2396e-05 | 0.0025 | 9.0756e-06 |
| | $Error_{(Last)}$ | 9.2396e-05 | 0.0025 | 7.9060e-06 |
| | $Time_{(PT)}$ | 0e+0 | 0e+0 | 0e+0 |
| 0.025 | $Error_{(Max)}$ | 2.2447e-05 | 0.0012 | 1.2961e-06 |
| | $Error_{(Last)}$ | 2.2447e-05 | 0.0012 | 1.2161e-06 |
| | $Time_{(PT)}$ | 0e+0 | 0e+0 | 0e+0 |
| 0.0125 | $Error_{(Max)}$ | 5.5374e-06 | 6.2417e-04 | 1.7074e-07 |
| | $Error_{(Last)}$ | 5.5374e-06 | 6.2411e-04 | 1.6330e-07 |
| | $Time_{(PT)}$ | 0e+0 | 0e+0 | 0e+0 |




In Fig (1-4) we display the graphs of three numerical schemes among them two well-known schemes named in abbreviation as RK3MH [7] and RK3MA [5], and one new Integrator Method. In every figure, x-values are spotted in x-axis and Log (Absolute Error) in y-axis. Among all graphically results the graph of new Integrator method shows less error than other methods whose graphical result present in graph. It is clearly observed that new Integrator Method is better numerically and converges rapidly in comparison with RK3MH and RK3MA to the exact solution.

Online ISSN 3007-3197
Print ISSN 3007-3189
http://amresearchreview.com/index.php/Journal/about

## CONCLUSION

In this Paper, New third order well-organized numerical technique for IVP of ODE's including partial derivative has constructed and enhanced its competency regarding truncation error. Taylor's series is utilized to derive and develop new family of numerical iterative integrative. Numerically and graphically, new proposed method named as integrator is examined for better to better consequences. Consequences (graphically and numerically) gives evidence that proposed scheme is more accurate, converge rapidly and competent than other in term of approximations. Absolute Errors (last and maximum) with CPU time is numerically obtained by using altered step size for comparison with existing schemes. The RK3MH [7] and RK3MA [5] rapidly grows in error than proposed one. According to the numerical result, newly developed accelerated method has accuracy of order three and keep nature explicit. We finally on that result the accelerated proposed method is more capable and appropriate.

## REFERENCES

[1] Ashiribo Senapon Wusu, Moses Adebowale Akanbi, and Solomon Adebola Okunuga, "A Three-Stage Multiderivative Explicit Runge-Kutta Method," *American Journal of Computational Mathematics*, vol. 3, pp. 121–126, 2013.

[2] J. C. Butcher, *Numerical Methods for Ordinary Differential Equations*. John Wiley & Sons, 2016.

[3] A. O. Owolanke, O. Uwaheren, and F. O. Obarhua, "An Eight Order Two-Step Taylor Series Algorithm for the Numerical Solutions of Initial Value Problems of Second Order Ordinary Differential Equations," *Open Access Library Journal*, vol. 4, p. E3486, 2017.

[4] D. G. Zill and W. S. Wright, *Differential Equations with Boundary-Value Problems*, 8th ed.

[5] A Third Order Runge-Kutta Method Based on a Linear Combination of Arithmetic Mean, Harmonic Mean and Geometric Mean.

[6] M. O. Turacı and T. Öziş, "Derivation of Three-Derivative Runge-Kutta Methods," *Springer Science Business Media New York*, 2016.

[7] E. Kreyszig, H. Kreyszig, and E. J. Norminton, *Advanced Engineering Mathematics*, 10th ed., 2011.

[8] A. M. Wazwaz, "A Modified Third Order Runge-Kutta Method," *Applied*

Online ISSN 3007-3197
Print ISSN 3007-3189
http://amresearchreview.com/index.php/Journal/about

*Mathematics Letters*, vol. 3, no. 3, pp. 123–125, 1990.

[9] R. V. Dukkipati, *Numerical Methods*, 1st ed., 2010.

[10] R. L. Burden and J. D. Faires, *Numerical Analysis*. Brooks/Cole, 2001.

[11] P. V. O'Neil, *Advanced Engineering Mathematics*, 7th ed.

[12] J. C. Butcher, "On Runge-Kutta Processes of High Order," *Journal of the Australian Mathematical Society*, vol. 4, 1964.

[13] S. O. Fatunla, *Numerical Methods for Initial Value Problems in Ordinary Differential Equations*. Academic Press, 1991.

[14] N. Ahmed, "Study of Numerical Accuracy of Runge-Kutta Second, Third and Fourth Order Method."

[15] E. Hairer, S. P. Nørsett, and G. Wanner, *Solving Ordinary Differential Equations I: Nonstiff Problems*, 2nd ed., Springer, 1993.

[16] W. Hundsdorfer and J. G. Verwer, *Numerical Solution of Time-Dependent Advection-Diffusion-Reaction Equations*. Springer, 2003.

[17] A. Kaw and E. Kalu, *Numerical Methods with Applications*, 2008.

[18] D. E. Dumitras, "Runge-Kutta-Fehlberg Type Procedure on Two Nodes for Numerical Integration of Systems of Differential Equations," *Automat. Comput. Appl. Math.*, vol. 2, pp. 139–143.

[19] W. H. Enright and T. E. Hull, "Test Results on Initial Value Methods for Non-Stiff Ordinary Differential Equations," *SIAM Journal on Numerical Analysis*, vol. 13, no. 6, pp. 944–961, 1976.

[20] J. H. Verner, "Some Runge-Kutta Formula Pairs," *SIAM Journal on Numerical Analysis*, vol. 28, no. 2, pp. 496–511, 1991.