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Significant Behavior of Generalized Fractional Operators with Bessel k-function and their Consequences

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ABSTRACT

In this paper, we discuss the generalized behavior of fractional operators with series type of special functions. The Saigo's generalized *k*-fractional integral operators involving *k*hypergeometric function as its kernel and its applications are also under discussion. Moreover, we resolve some results of generalized fractional integral operators with power function in terms of *k* are solved. Such types of results are represented in terms of generalized *k*-Wright hypergeometric functions. We also discuss the behavior of fractional operators with *k*-Bessel function and further in terms of generalized *k*-hypergeometric function. All the results obtained have immense applications in the field of analysis of operator theory as well as special functions in mathematics.

Keywords: generalized *k*-fractional integral operators; *k*-hypergeometric function; generalized *k*-Wright hypergeometric function; *k*-Bessel function of the first kind; generalized *k*-hypergeometric function, **26A33; 33C05; 33C10; 33C20; 33C60; 26A**

1 Introduction

Fractional calculus, which extends the classical notions of differentiation and integration to non-integer orders, has become an indispensable tool in modern mathematical analysis due to its capability of modeling memory and hereditary effects. Such properties are essential in the mathematical formulation of physical, engineering, and biological processes governed by nonlocal dynamics [8–10]. As a result, fractional integral and differential operators have been widely studied from both theoretical and applied perspectives.

Among the classical fractional operators, the Riemann–Liouville fractional integral of order $\alpha > 0$ is defined by

$$(I_{a^+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \quad x > a, \tag{1.1}$$

where $\Gamma(\cdot)$ denotes the Euler gamma function [8]. Despite its importance, the classical Riemann–Liouville operator is sometimes insufficient to describe complex systems. This limitation has motivated the development of generalized fractional operators involving special functions such as the Mittag–Leffler and Prabhakar functions [11].

Parallel to these developments, the theory of k-calculus has emerged as a natural generalization of classical calculus. The k-gamma function, introduced to extend the classical gamma function, is defined by [12]

$$\Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{-t^k/k}\, dt, \quad k > 0, \tag{1.2}$$

and satisfies the relation $\Gamma_k(x+k) = x\Gamma_k(x)$. Using $\Gamma_k(\cdot)$, the k-Pochhammer symbol is given by

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k), \quad n \in \mathbb{N}. \tag{1.3}$$

An important special function arising from k-series is the k-Bessel function of the first kind, defined by [12, 13]

$$J_{\nu}^{k}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma_k(nk+\nu+k)\, n!} \left(\frac{x}{2}\right)^{2n+\nu}, \quad \nu > -k. \tag{1.4}$$

For $k = 1$, this function reduces to the classical Bessel function of the first kind, showing that the k-Bessel function provides a genuine generalization.

Recently, generalized fractional integral operators involving k-series kernels have attracted considerable interest. A typical generalized fractional integral operator associated with a k-special function can be written in the form

$$(\mathcal{I}_{a^+}^{\alpha,k} f)(x) = \int_a^x (x-t)^{\alpha-1} \Phi_k(\lambda(x-t)^{\mu})\, f(t)\, dt, \tag{1.5}$$

where $\Phi_k(\cdot)$ represents a suitable k-series function, such as the k-Mittag–Leffler or k-Bessel function, and $\alpha, \mu, \lambda > 0$.

Integral inequalities involving such operators play a crucial role in mathematical analysis. For a convex function $f : [a, b] \to \mathbb{R}$, the classical Hermite–Hadamard inequality states that

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.6}$$

This inequality has been extended to fractional integrals and further generalized using k-fractional operators [14, 15]. In particular, inequalities of the type

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)^\alpha}\left[(\mathcal{I}^{\alpha,k}_{a^+}f)(b) + (\mathcal{I}^{\alpha,k}_{b^-}f)(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.7}$$

have been obtained under suitable assumptions on f and the kernel Φ_k.

Motivated by these recent advances, the present paper is devoted to establishing **some new results for generalized fractional integral operators involving k-series**, with particular emphasis on kernels defined via the **k-Bessel function of the first kind**. Several new integral inequalities are derived, extending and unifying existing results in the literature. The generality of the proposed framework is illustrated by showing that many known inequalities follow as special cases for appropriate choices of parameters.

2 Preliminaries

In this section, we derive the fundamental results of left and right sided generalized fractional integration of a power function at level k in the form of lemmas.

Lemma 2.1. *Let $\alpha', \beta', \eta' \in \mathbb{C}$ then there holds the relation*

$$(I^{\alpha',\beta',\eta'}_{0,y}s^{\sigma'-1})_k(y) = y^{\frac{\sigma'-\beta'}{k}-1}\frac{\Gamma_k(\sigma')\Gamma_k(\alpha'+\sigma'-\alpha'k+\eta'-\beta'k)}{\Gamma_k(\alpha'+\sigma'-\alpha'k-\beta'k)\Gamma_k(\sigma'+\alpha'+\eta')}. \tag{2.1}$$

Proof. Consider the left sided generalized k-fractional integral operator

$$(I^{\alpha',\beta',\eta'}_{0,y}g)_k(y) = \frac{y^{\frac{-\alpha'-\beta'}{k}}}{k\Gamma_k(\alpha')}\int_0^y (y-s)^{\frac{\alpha'}{k}-1}$$
$$\times\, {}_2F_{1,k}((\alpha'+\beta',k),(-\eta',k);(\alpha',k);1-\frac{s}{y})g(s)ds. \tag{2.2}$$

Using the equation (**??**) and k-power function in equation (2.2), we get

$$(I^{\alpha',\beta',\eta'}_{0,y}s^{\sigma'-1})_k(y) = \frac{y^{\frac{-\alpha'-\beta'}{k}}}{k\Gamma_k(\alpha')}\int_0^y (y-s)^{\frac{\alpha'}{k}-1}$$
$$\times \sum_{m=0}^{\infty}\frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k}m!}(1-\frac{s}{y})^m s^{\frac{\sigma'}{k}-1}ds. \tag{2.3}$$

By putting

$$s = vy \implies ds = ydv$$

$$s = 0 \implies v = 0$$
$$s = y \implies v = 1$$

in equation (2.3), we obtain

$$(I_{0,y}^{\alpha',\beta',\eta'} s^{\sigma'-1})_k(y) = \frac{y^{\frac{\sigma'-\beta'}{k}-1}}{\Gamma_k(\alpha')} \sum_{m=0}^{\infty} \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k} m!}$$
$$\times \frac{1}{k}\int_0^1 (1-v)^{\frac{\alpha'+mk}{k}-1} v^{\frac{\sigma'}{k}-1} dv. \quad (2.4)$$

Since

$$\beta_k(l,h) = \frac{\Gamma_k(l)\Gamma_k(h)}{\Gamma_k(l+h)}. \quad (2.5)$$

By using the equation (**??**) and equation (2.5) in equation (2.4), we have

$$(I_{0,y}^{\alpha',\beta',\eta'} s^{\sigma'-1})_k(y) = \frac{y^{\frac{\sigma'-\beta'}{k}-1}}{\Gamma_k(\alpha')} \sum_{m=0}^{\infty} \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k} m!} \frac{\Gamma_k(\alpha'+mk)\Gamma_k(\sigma')}{\Gamma_k(\alpha'+\sigma'+mk)}. \quad (2.6)$$

Since

$$\Gamma_k(t+mk) = (t)_{m,k}\Gamma_k(t). \quad (2.7)$$

Using the equation (2.7) in equation (2.6), we get

$$(I_{0,y}^{\alpha',\beta',\eta'} s^{\sigma'-1})_k(y) = y^{\frac{\sigma'-\beta'}{k}-1} \frac{\Gamma_k(\sigma')}{\Gamma_k(\alpha'+\sigma')} \sum_{m=0}^{\infty} \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha'+\sigma')_{m,k} m!}. \quad (2.8)$$

Using the equation (**??**) in equation (2.8), we have

$$(I_{0,y}^{\alpha',\beta',\eta'} s^{\sigma'-1})_k(y) = y^{\frac{\sigma'-\beta'}{k}-1} \frac{\Gamma_k(\sigma')}{\Gamma_k(\alpha'+\sigma')} {}_2F_{1,k}((\alpha'+\beta',k),(-\eta',k);(\alpha'+\sigma',k);1).$$

It can also be written as

$$(I_{0,y}^{\alpha',\beta',\eta'} s^{\sigma'-1})_k(y) = y^{\frac{\sigma'-\beta'}{k}-1} \frac{\Gamma_k(\sigma')}{\Gamma_k(\alpha'+\sigma')}$$
$$\times {}_2F_{1,k}((\alpha'+\beta',1),(-\eta',k);(\alpha'+\sigma',k);1). \quad (2.9)$$

Since in [7],

$$21,k(\alpha',1),(\beta',k)(\eta',k)1 = \frac{\Gamma_k(\eta')\Gamma_k(\eta'-\beta'-k\alpha')}{\Gamma_k(\eta'-k\alpha')\Gamma_k(\eta'-\beta')}. \quad (2.10)$$

Using the equation (2.10) in equation (2.9), we obtain

$$(I_{0,y}^{\alpha',\beta',\eta'} s^{\sigma'-1})_k(y) = y^{\frac{\sigma'-\beta'}{k}-1} \frac{\Gamma_k(\sigma')\Gamma_k(\alpha'+\sigma'+\eta'-\alpha'k-\beta'k)}{\Gamma_k(\alpha'+\sigma'-\alpha'k-\beta'k)\Gamma_k(\sigma'+\alpha'+\eta')}.$$

□

Lemma 2.2. *Let $\alpha', \beta', \eta' \in \mathbb{C}$ then there holds the relation*

$$(I_{y,\infty}^{\alpha',\beta',\eta'} s^{\sigma'-1})_k(y) = y^{\frac{\sigma'-\beta'}{k}-1} \times \frac{\Gamma_k(\beta'-\sigma'+k)\Gamma_k(\alpha'+\beta'+\eta'-\alpha'k-\beta'k-\sigma'+k)}{\Gamma_k(\alpha'+\beta'+k-\sigma'-\alpha'k-\beta'k)\Gamma_k(\alpha'+\beta'+\eta'-\sigma'+k)}. \quad (2.11)$$

Proof. Consider the right sided generalized k-fractional integral operator

$$(I_{y,\infty}^{\alpha',\beta',\eta'} g)_k(y) = \frac{1}{k\Gamma_k(\alpha')} \int_y^\infty (s-y)^{\frac{\alpha'}{k}-1} s^{\frac{-\alpha'-\beta'}{k}} \times {}_2F_{1,k}((\alpha'+\beta',k),(-\eta',k);(\alpha',k);1-\frac{y}{s}) g(s) ds. \quad (2.12)$$

By apply k-power function in equation (2.12), we get

$$(I_{y,\infty}^{\alpha',\beta',\eta'} s^{\sigma'-1})_k(y) = \frac{1}{k\Gamma_k(\alpha')} \int_y^\infty (s-y)^{\frac{\alpha'}{k}-1} s^{\frac{-\alpha'-\beta'}{k}} \times \sum_{m=0}^{\infty} \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k} m!} (1-\frac{y}{s})^m s^{\frac{\sigma'}{k}-1} ds. \quad (2.13)$$

By putting

$$s = \frac{y}{v} \implies ds = -\frac{y}{v^2} dv \quad (2.14)$$
$$s = y \implies v = 1 \quad (2.15)$$
$$s = \infty \implies v = 0$$

in equation (2.13), we obtain

$$(I_{y,\infty}^{\alpha',\beta',\eta'} s^{\sigma'-1})_k(y) = \frac{y^{\frac{\sigma'-\beta'}{k}-1}}{\Gamma_k(\alpha')} \sum_{m=0}^{\infty} \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k} m!} \times \frac{1}{k} \int_0^1 (1-v)^{\frac{\alpha'+mk}{k}-1} v^{\frac{\beta'-\sigma'}{k}+1-1} dv. \quad (2.16)$$

After simplification, we have

$$(I_{y,\infty}^{\alpha',\beta',\eta'} s^{\sigma'-1})_k(y) = \frac{y^{\frac{\sigma'-\beta'}{k}-1}}{\Gamma_k(\alpha')} \times \sum_{m=0}^{\infty} \frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha')_{m,k} m!} \frac{\Gamma_k(\alpha'+mk)\Gamma_k(\beta'-\sigma'+k)}{\Gamma_k(\alpha'+\beta'-\sigma'+mk+k)}. \quad (2.17)$$

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Using the equation (2.7) in equation (2.17), we obtain

$$(I_{y,\infty}^{\alpha',\beta',\eta'}s^{\sigma'-1})_k(y) = y^{\frac{\sigma'-\beta'}{k}-1}\frac{\Gamma_k(\beta'-\sigma'+k)}{\Gamma_k(\alpha'-\sigma'+\beta'+k)}\sum_{m=0}^{\infty}\frac{(\alpha'+\beta')_{m,k}(-\eta')_{m,k}}{(\alpha'-\sigma'+\beta'+k)_{m,k}m!}. \quad (2.18)$$

After simplify equation (2.18), we have

$$(I_{y,\infty}^{\alpha',\beta',\eta'}s^{\sigma'-1})_k(y) = y^{\frac{\sigma'-\beta'}{k}-1}\frac{\Gamma_k(\beta'-\sigma'+k)}{\Gamma_k(\alpha'-\sigma'+\beta'+k)} \times {}_2F_{1,k}((\alpha'+\beta',1),(-\eta',k);(\alpha'-\sigma'+\beta'+k,k);1). \quad (2.19)$$

Using the equation (2.10) in equation (2.19), we get

$$(I_{y,\infty}^{\alpha',\beta',\eta'}s^{\sigma'-1})_k(y) = y^{\frac{\sigma'-\beta'}{k}-1}\frac{\Gamma_k(\beta'-\sigma'+k)\Gamma_k(\alpha'-\sigma'+\beta'+k-\alpha'k-\beta'k+\eta')}{\Gamma_k(\alpha'-\sigma'+\beta'+k-\alpha'k-\beta'k)\Gamma_k(\alpha'-\sigma'+\beta'+k+\eta')}.$$

□

3 Representation in terms of generalized k-Wright hypergeometric function

In this section, we solve generalized left and right sided generalized k-fractional integration for the k-Bessel functions of the first kind, that is represented in terms of generalized k-Wright hypergeometric function.

Theorem 3.1. *Let $\alpha', \beta', \eta', \sigma', v \in \mathbb{C}$ then there holds the formula*

$$(I_{0,y}^{\alpha',\beta',\eta'}s^{\sigma'-1}J_v(s))_k(y) = \frac{y^{\frac{\sigma'+v-\beta'}{k}-1}}{2^{\frac{v}{k}}} \times {}_2\Psi_3^k\left[\begin{matrix}(\sigma'+v,2),(\alpha'+\sigma'+\eta'+v-\alpha'k-\beta'k,2)\\(\alpha'+\sigma'+v-\alpha'k-\beta'k,2),(\sigma'+v+\alpha'+\eta',2),(v+k,1)\end{matrix}\middle| -\frac{y^2}{4}\right]. \quad (3.1)$$

Proof. Using k-power function and the equation (**??**) in left sided generalized k-fractional integral operator, we have

$$(I_{0,y}^{\alpha',\beta',\eta'}s^{\sigma'-1}J_v(s))_k(y) = \sum_{m=0}^{\infty}\frac{(-1)^m(\frac{1}{2})^{\frac{v}{k}+2m}}{\Gamma_k(v+mk+k)m!}(I_{0,y}^{\alpha',\beta',\eta'}s^{v+\sigma'+2mk-1})_k(y). \quad (3.2)$$

Using the lemma (3.1.1) as a result and using the equation (2.1) with σ' replaced by $\sigma'+v+2mk$ in equation (3.2), we obtain

$$(I_{0,y}^{\alpha',\beta',\eta'}s^{\sigma'-1}J_v(s))_k(y) = \sum_{m=0}^{\infty}\frac{(-1)^m(\frac{1}{2})^{\frac{v}{k}+2m}}{\Gamma(v+mk+k)m!}y^{\frac{\sigma'+v+2mk}{k}-\frac{\beta'}{k}-1} \times \frac{\Gamma_k(\sigma'+v+2mk)\Gamma_k(\alpha'+\sigma'+v+2mk-\alpha'k-\beta'k+\eta')}{\Gamma_k(\alpha'+\sigma'+v+2mk-\alpha'k-\beta'k)\Gamma_k(\sigma'+v+2mk+\alpha'+\eta')}$$

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$$= \frac{y^{\frac{\sigma'+\nu-\beta'}{k}-1}}{2^{\frac{\nu}{k}}}$$
$$\times \sum_{m=0}^{\infty} \frac{\Gamma_k(\nu+\sigma'+2mk)\Gamma_k(\alpha'+\nu+\sigma'+\eta'-\alpha'k-\beta'k+2mk)}{\Gamma_k(\alpha'+\sigma'+\nu-\alpha'k-\beta'k+2mk)\Gamma_k(\sigma'+\nu+\alpha'+\eta'+2mk)\Gamma_k(\nu+k+mk)}$$
$$\times \frac{(-\frac{y^2}{4})^m}{m!}. \quad (3.3)$$

After simplification, we have

$$(I_{0,y}^{\alpha',\beta',\eta'} s^{\sigma'-1} J_\nu(s))_k(y) = \frac{y^{\frac{\sigma'+\nu-\beta'}{k}-1}}{2^{\frac{\nu}{k}}}$$
$$\times {}_2\Psi_3^k \left[\begin{matrix} (\sigma'+\nu,2),(\alpha'+\sigma'+\eta'+\nu-\alpha'k-\beta'k,2) \\ (\alpha'+\sigma'+\nu-\alpha'k-\beta'k,2),(\sigma'+\nu+\alpha'+\eta',2),(\nu+k,1) \end{matrix} \middle| -\frac{y^2}{4} \right].$$

□

Theorem 3.2. *Let $\alpha', \beta', \eta', \sigma', \nu \in \mathbb{C}$ then there holds the formula*

$$(I_{y,\infty}^{\alpha',\beta',\eta'} t^{\sigma'-1} J_\nu(\frac{1}{s}))_k(y) = \frac{y^{\frac{\sigma'-\nu-\beta'}{k}-1}}{2^{\frac{\nu}{k}}} \quad (3.4)$$

$$\times {}_2\Psi_3^k \left[\begin{matrix} (k+\beta'-\sigma'+\nu,2),(k+\eta'+\alpha'+\beta'-\sigma'+\nu-\alpha'k-\beta'k,2) \\ (\alpha'+\beta'+k-\sigma'+\nu-\alpha'k-\beta'k,2),(k+\beta'+\alpha'+\eta'-\sigma'+\nu,2),(\nu+k,1) \end{matrix} \middle| \frac{-1}{4y^2} \right].$$

Proof. Using k-power function and the equation (**??**) in right sided generalized k-fractional integral operator, we have

$$(I_{y,\infty}^{\alpha',\beta',\eta'} s^{\sigma'-1} J_\nu(\frac{1}{s}))_k(y) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2})^{\frac{\nu}{k}+2m}}{\Gamma_k(\nu+mk+k)m!} (I_{y,\infty}^{\alpha',\beta',\eta'} s^{\sigma'-\nu-2mk-1})_k(y). \quad (3.5)$$

Using the lemma (3.1.2) and using equation (2.11) with σ' replaced by $\sigma'-\nu-2mk$ in equation (3.5), we obtain

$$(I_{y,\infty}^{\alpha',\beta',\eta'} s^{\sigma'-1} J_\nu(\tfrac{1}{s}))_k(y) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2})^{\frac{\nu}{k}+2m}}{\Gamma_k(\nu+mk+k)m!} y^{\frac{\sigma'-\nu-2mk-\beta'}{k}-1}$$
$$\times \frac{\Gamma_k(\beta'-\sigma'+\nu+k+2mk)\Gamma_k(\alpha'+\beta'+\eta'-\sigma'+\nu-\alpha'k-\beta'k+k+2mk)}{\Gamma_k(\alpha'+\beta'+k-\sigma'+\nu-\alpha'k-\beta'k+2mk)\Gamma_k(\alpha'+\beta'+\eta'-\sigma'+\nu+k+2mk)}.$$

$$= \frac{y^{\frac{\sigma'-\nu-\beta'}{k}-1}}{2^{\frac{\nu}{k}}}$$
$$\times \sum_{m=0}^{\infty} \frac{\Gamma_k(\beta'-\sigma'+\nu+k+2mk)\Gamma_k(\eta'+\alpha'+\beta'-\sigma'+\nu-\alpha'k-\beta'k+k+2mk)}{\Gamma_k(\alpha'+\beta'+k-\sigma'+\nu-\alpha'k-\beta'k+2mk)\Gamma(\alpha'+\beta'+\eta'-\sigma'+\nu+k+2mk)}$$

$$\times \frac{1}{\Gamma(\nu+k+mk)} \frac{(-\frac{1}{4y^2})^m}{m!}. \quad (3.6)$$

After simplification, we have

$$(I_{y,\infty}^{\alpha',\beta',\eta'} t^{\sigma'-1} J_\nu(\frac{1}{s}))_k(y) = \frac{y^{\frac{\sigma'-\nu-\beta'}{k}-1}}{2^{\frac{\nu}{k}}}$$

$$\times {}_2\Psi_3^k \left[\begin{matrix} (k+\beta'-\sigma'+\nu,2),(k+\eta'+\alpha'+\beta'-\sigma'+\nu-\alpha'k-\beta'k,2) \\ (\alpha'+\beta'+k-\sigma'+\nu-\alpha'k-\beta'k,2),(k+\beta'+\alpha'+\eta'-\sigma'+\nu,2),(\nu+k,1) \end{matrix} \middle| \frac{-1}{4y^2} \right].$$

$\square$

4 Representation in terms of generalized k-hypergeometric function

In this section, we represent equation (3.1) and equation (3.4) in terms of generalized k-hypergeometric function.

Theorem 4.1. *Let $\alpha', \beta', \eta', \sigma', \nu \in \mathbb{C}$ then holds the formula*

$$(I_{0,y}^{\alpha',\beta',\eta'} s^{\sigma'-1} J_\nu(s))_k(y) = \frac{y^{\frac{\sigma'+\nu-\beta'}{k}-1}}{2^{\frac{\nu}{k}}}$$

$$\times \frac{\Gamma_k(\sigma'+\nu)\Gamma_k(\alpha'+\sigma'+\nu+\eta'-\alpha'k-\beta'k)}{\Gamma_k(\alpha'+\sigma'+\nu-\alpha'k-\beta'k)\Gamma_k(\sigma'+\nu+\alpha'+\eta')\Gamma(\nu+k)} \quad (4.1)$$

$$\times {}_4F_{5,k}(\tfrac{\sigma'+\nu}{2},k),(\tfrac{\sigma'+\nu+k}{2},k),(\tfrac{\alpha'+\sigma'+\nu+\eta'-\alpha'k-\beta'k}{2},k),(\tfrac{\alpha'+\sigma'+\nu+\eta'-\alpha'k-\beta'k+k}{2},k)(\nu+k,k),(\tfrac{\alpha'+\sigma'+\nu-\alpha'k-\beta'}{2}$$

$$\times {}_4F_{5,k} \left[\begin{matrix} (\frac{\sigma'+\nu}{2},k),(\frac{\sigma'+\nu+k}{2},k),(\frac{\alpha'+\sigma'+\nu+\eta'-\alpha'k-\beta'k}{2},k),(\frac{\alpha'+\sigma'+\nu+\eta'-\alpha'k-\beta'k+k}{2},k) \\ (\nu+k,k),(\frac{\alpha'+\sigma'+\nu-\alpha'k-\beta'k}{2},k),(\frac{\alpha'+\sigma'+\nu-\alpha'k-\beta'k+k}{2},k),(\frac{\sigma'+\nu+\alpha'+\eta'}{2},k),(\frac{\sigma'+\nu+\alpha'+\eta'+k}{2},k) \end{matrix} \middle| \frac{-1}{4y^2} \right].$$

Proof. Using the equation (2.7) in equation (3.3), we get

$$(I_{0,y}^{\alpha',\beta',\eta'} s^{\sigma'-1} J_\nu(s))_k(y) = \frac{y^{\frac{\sigma'+\nu-\beta'}{k}-1}}{2^{\frac{\nu}{k}}} \frac{\Gamma_k(\nu+\sigma')\Gamma_k(\alpha'+\nu+\sigma'+\eta'-\alpha'k-\beta'k)}{\Gamma_k(\alpha'+\sigma'+\nu-\alpha'k-\beta'k)\Gamma_k(\sigma'+\nu+\alpha'+\eta')\Gamma_k(\nu+k)}$$

$$\times \sum_{m=0}^{\infty} \frac{(\nu+\sigma')_{2m,k}(\alpha'+\nu+\sigma'+\eta'-\alpha'k-\beta'k)_{2m,k}}{(\alpha'+\sigma'+\nu-\alpha'k-\beta'k)_{2m,k}(\sigma'+\nu+\alpha'+\eta')_{2m,k}(\nu+k)_{m,k}} \frac{(-\frac{y^2}{4})^m}{m!}. \quad (4.2)$$

Since

$$(t)_{2m,k} = 2^{2m}(\frac{t}{2})_{m,k}(\frac{t+k}{2})_{m,k}. \quad (4.3)$$

Using the equation (4.3) in equation (4.2), we obtain

$$(I_{0,y}^{\alpha',\beta',\eta'} s^{\sigma'-1} J_\nu(s))_k(y) = \frac{y^{\frac{\sigma'+\nu-\beta'}{k}-1}}{2^{\frac{\nu}{k}}} \frac{\Gamma_k(\nu+\sigma')\Gamma_k(\alpha'+\nu+\sigma'+\eta'-\alpha'k-\beta'k)}{\Gamma_k(\alpha'+\sigma'+\nu-\alpha'k-\beta'k)\Gamma_k(\sigma'+\nu+\alpha'+\eta')\Gamma_k(\nu+k)}$$
$$\times \sum_{m=0}^{\infty} \frac{(\frac{\nu+\sigma'}{2})_{m,k}(\frac{\nu+\sigma'+k}{2})_{m,k}(\frac{\alpha'+\nu+\sigma'+\eta'-\alpha'k-\beta'k}{2})_{m,k}(\frac{\alpha'+\nu+\sigma'+\eta'-\alpha'k-\beta'k+k}{2})_{m,k}}{(\frac{\alpha'+\sigma'+\nu-\alpha'k-\beta'k}{2})_{m,k}(\frac{\alpha'+\sigma'+\nu-\alpha'k-\beta'k+k}{2})_{m,k}(\frac{\sigma'+\nu+\alpha'+\eta'}{2})_{m,k}(\frac{\sigma'+\nu+\alpha'+\eta'+k}{2})_{m,k}(\nu+k)_{m,k}}$$
$$\times \frac{(-\frac{y^2}{4})^m}{m!}. \quad (4.4)$$

After simplification, we have

$$(I_{0,y}^{\alpha',\beta',\eta'} s^{\sigma'-1} J_\nu(s))_k(y) = \frac{y^{\frac{\sigma'+\nu-\beta'}{k}-1}}{2^{\frac{\nu}{k}}}$$
$$\times \frac{\Gamma_k(\sigma'+\nu)\Gamma_k(\alpha'+\sigma'+\nu+\eta'-\alpha'k-\beta'k)}{\Gamma_k(\alpha'+\sigma'+\nu-\alpha'k-\beta'k)\Gamma_k(\sigma'+\nu+\alpha'+\eta')\Gamma(\nu+k)}$$
$$\times {}_4F_{5,k}\left[\begin{matrix} (\frac{\sigma'+\nu}{2},k),(\frac{\sigma'+\nu+k}{2},k),(\frac{\alpha'+\sigma'+\nu+\eta'-\alpha'k-\beta'k}{2},k),(\frac{\alpha'+\sigma'+\nu+\eta'-\alpha'k-\beta'k+k}{2},k) \\ (\nu+k,k),(\frac{\alpha'+\sigma'+\nu-\alpha'k-\beta'k}{2},k),(\frac{\alpha'+\sigma'+\nu-\alpha'k-\beta'k+k}{2},k),(\frac{\sigma'+\nu+\alpha'+\eta'}{2},k),(\frac{\sigma'+\nu+\alpha'+\eta'+k}{2},k) \end{matrix} \middle| \frac{-1}{4y^2}\right]. \quad \square$$

Theorem 4.2. *Let $\alpha', \beta', \eta', \sigma', \nu \in \mathbb{C}$ then holds the formula*

$$(I_{y,\infty}^{\alpha',\beta',\eta'} s^{\sigma'-1} J_\nu(\tfrac{1}{s}))_k(y) = \frac{y^{\frac{\sigma'-\nu-\beta'}{k}-1}}{2^{\frac{\nu}{k}}} \frac{\Gamma_k(\beta'-\sigma'+\nu+k)}{\Gamma_k(k-\sigma'+\nu+\alpha'+\beta'-\alpha'k-\beta'k)}$$
$$\times \frac{\Gamma_k(\alpha'+\beta'+\eta'-\sigma'+\nu-\alpha'k-\beta'k+k)}{\Gamma_k(\alpha'+\beta'+\eta'-\sigma'+\nu+k)\Gamma_k(\nu+k)} \quad (4.5)$$

$$\times {}_4F_{5,k}(\frac{\beta'-\sigma'+\nu+k}{2},k),(\frac{\beta'-\sigma'+\nu+2k}{2},k),(\frac{\alpha'+\beta'+\eta'-\sigma'+\nu-\alpha'k-\beta'k+k}{2},k),(\frac{\alpha'+\beta'+\eta'-\sigma'+\nu-\alpha'k-\beta'k+2k}{2},k);(\nu+k,k),(\frac{k}{}$$
$$\times {}_4F_{5,k}\left[\begin{matrix} (\frac{\beta'-\sigma'+\nu+k}{2},k),(\frac{\beta'-\sigma'+\nu+2k}{2},k),(\frac{\alpha'+\beta'+\eta'-\sigma'+\nu-\alpha'k-\beta'k+k}{2},k),(\frac{\alpha'+\beta'+\eta'-\sigma'+\nu-\alpha'k-\beta'k+2k}{2},k) \\ (\nu+k,k),(\frac{k-\sigma'+\nu+\alpha'+\beta'-\alpha'k-\beta'k}{2},k),(\frac{2k-\sigma'+\nu+\alpha'+\beta'-\alpha'k-\beta'k}{2},k),(\frac{\alpha'+\beta'+\eta'-\sigma'+\nu+k}{2},k),(\frac{\alpha'+\beta'+\eta'-\sigma'+\nu+2k}{2},k) \end{matrix} \middle| \frac{-1}{4y^2}\right].$$

Proof. Using the equation (2.7) in equation (3.6), we get

$$(I_{y,\infty}^{\alpha',\beta',\eta'} s^{\sigma'-1} J_\nu(\tfrac{1}{s}))_k(y) = \frac{y^{\frac{\sigma'-\nu-\beta'}{k}-1}}{2^{\frac{\nu}{k}}} \frac{\Gamma_k(\beta'-\sigma'+\nu+k)\Gamma_k(\alpha'+\beta'+\eta'-\sigma'+\nu-\alpha'k-\beta'k+k)}{\Gamma_k(k-\sigma'+\alpha'+\beta'+\nu-\alpha'k-\beta'k)\Gamma_k(\alpha'+\beta'+\eta'-\sigma'+\nu+k)\Gamma_k(\nu+k)}$$
$$\times \sum_{m=0}^{\infty} \frac{(\beta'-\sigma'+\nu+k)_{2m,k}(\alpha'+\beta'+\eta'-\sigma'+\nu-\alpha'k-\beta'k+k)_{2m,k}}{(k-\sigma'+\nu+\alpha'+\beta'-\alpha'k-\beta'k)_{2m,k}(\alpha'+\beta'+\eta'-\sigma'+\nu+k)_{2m,k}(\nu+k)_{m,k}}$$
$$\times \frac{(-\frac{1}{4y^2})^m}{m!}. \quad (4.6)$$

Using the equation (4.3) in equation (4.6), we obtain

$$(I_{y,\infty}^{\alpha',\beta',\eta'} s^{\sigma'-1} J_\nu(\tfrac{1}{s}))_k(y) = \frac{y^{\frac{\sigma'-\nu-\beta'}{k}-1}}{2^{\frac{\nu}{k}}} \frac{\Gamma_k(\beta'-\sigma'+\nu+k)\Gamma_k(\alpha'+\beta'+\eta'-\sigma'+\nu-\alpha'k-\beta'k+k)}{\Gamma_k(k-\sigma'+\nu+\alpha'+\beta'-\alpha'k-\beta'k)\Gamma_k(\alpha'+\beta'+\eta'-\sigma'+\nu+k)\Gamma_k(\nu+k)}$$
$$\times \sum_{m=0}^{\infty} \frac{(\frac{\beta'-\sigma'+\nu+k}{2})_{m,k}(\frac{\beta'-\sigma'+\nu+2k}{2})_{m,k}(\frac{\alpha'+\beta'+\eta'-\sigma'+\nu-\alpha'k-\beta'k+k}{2})_{m,k}}{(\frac{k-\sigma'+\nu+\alpha'+\beta'-\alpha'k-\beta'k}{2})_{m,k}(\frac{2k-\sigma'+\nu+\alpha'+\beta'-\alpha'k-\beta'k}{2})_{m,k}(\frac{\alpha'+\beta'+\eta'-\sigma'+\nu+k}{2})_{m,k}}$$
$$\times \frac{(\frac{\alpha'+\beta'+\eta'-\sigma'+\nu-\alpha'k-\beta'k+2k}{2})_{m,k}}{(\frac{\alpha'+\beta'+\eta'-\sigma'+\nu+2k}{2})_{m,k}(\nu+k)_{m,k}} \frac{(-\frac{1}{4y^2})^m}{m!}. \tag{4.7}$$

After simplification, we have

$$(I_{y,\infty}^{\alpha',\beta',\eta'} s^{\sigma'-1} J_\nu(\frac{1}{s}))_k(y) = \frac{y^{\frac{\sigma'-\nu-\beta'}{k}-1}}{2^{\frac{\nu}{k}}}$$
$$\times \frac{\Gamma_k(\beta'-\sigma'+\nu+k)\Gamma_k(\alpha'+\beta'+\eta'-\sigma'+\nu-\alpha'k-\beta'k+k)}{\Gamma_k(k-\sigma'+\nu+\alpha'+\beta'-\alpha'k-\beta'k)\Gamma_k(\alpha'+\beta'+\eta'-\sigma'+\nu+k)\Gamma_k(\nu+k)}$$
$$\times {}_4F_{5,k}\left[\begin{matrix} (\frac{\beta'-\sigma'+\nu+k}{2},k),(\frac{\beta'-\sigma'+\nu+2k}{2},k),(\frac{\alpha'+\beta'+\eta'-\sigma'+\nu-\alpha'k-\beta'k+k}{2},k),(\frac{\alpha'+\beta'+\eta'-\sigma'+\nu-\alpha'k-\beta'k+2k}{2},k) \\ (\nu+k,k),(\frac{k-\sigma'+\nu+\alpha'+\beta'-\alpha'k-\beta'k}{2},k),(\frac{2k-\sigma'+\nu+\alpha'+\beta'-\alpha'k-\beta'k}{2},k),(\frac{\alpha'+\beta'+\eta'-\sigma'+\nu+k}{2},k),(\frac{\alpha'+\beta'+\eta'-\sigma'+\nu+2k}{2},k)\end{matrix} \middle| \frac{-1}{4y^2}\right].$$

□

5 Conclusion

In This paper, we discussed the generalized fractional operators and its extension in k-series. Also, established the generalized version of Wright functions by implementation of generalized fractional k-operators having hypergeometric function as its kernel with power series k-functions. Moreover, we developed the composition of such type of generalized fractional operators with the new version of Bessel-function, and obtained results in term of Wright functions. We concluded that such type of operators generate the Wright functions, and also the composition of Bessel-functions and generalized fractional operators exist.

Declarations

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Data Availability Statement

Data sets were not generated or analyzed during the current study.

Conflict of Interest

The authors declare that they have no conflict of interest.

Clinical trial number

Not applicable.

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