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Finite Element Method (FEM) for Solving Time-Fractional Partial Differential Equations

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ABSTRACT

The critical, and effective Finite element Method (FEM) given in this paper can adequately trace the time-fractional of partial derivative equations (TFPDE) with the labels of factors of (Caputo) form of derivatives, incorporating the memory and hereditary influences in the majority of the physical and engineering frameworks. The study begins with setting the weak variational problem followed by building of the fully discrete FEM system regarding both spatial Galerkin discretization and two times strategies L1 quadrature approximation and Convolution Quadrature (CQBDF2) approach. Theoretical analyses based on Lax -Milgram lemma show the existence and uniqueness of weak solution and error and stability analysis and bolster the optimality of the convergence rates between O(h 2 + t 2): L1 O(h 2 + t 2): CQ-BDF2. Numerical experiments applied on benchmark problems which guarantee good accuracy, robust performance and efficiency even in conditions with weak singularity and heterogeneity are in support of such theoretical predictions. The capability of the framework to handle irregular geometries and changing coefficients is a testament to the fact that that framework can be used to simulate a wide variety of phenomena in the real world, such as anomalous diffusion, viscoelastic relaxation, and biological transport. The paper also indicates the future directions that can be enhanced in terms of adaptive mesh refinement, efficient time-stepping schemes and high-performance

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parallel implementation which will increase simulation of high-dimensional fractional systems as scalable.

Keywords: Finite Element Method (FEM), Time-Fractional Partial Differential Equations (TFPDEs), Caputo Derivative, Convolution Quadrature, L1 Scheme, Fractional Calculus, Stability Analysis, Error Estimates, Anomalous Diffusion, Computational Modeling.

Introduction

The use of time-fractional partial differential equations (TFPDEs) has emerged as a new basic mathematical tool to gain insight into the dynamic systems of interest with memory and anomalous diffusion behavior that cannot be elucidated using traditional integer-order modeling. These equations contain ideal examples of the application of the fractional derivatives to explain non-local dynamic process in time; this is significant because it implies that the current state of these systems not only exists in response to the current rate at which system operates, but it also depends on the past of the system (Podlubny, 1999; Kilbas et al., 2006). This memory dependence can be seen in the physical, biological and engineering uses of viscoelastic materials, heat conduction in heterogeneous materials,[2] subdiffusive transport through porous structure and conveyance of signals through biological tissues (Mainardi, 2010; Metzler and Klafer, 2000). It is the franchise, rather than specific paradigms, of the ratioed order modeling methodology, which offers rational coherent approach to model power law memory kernel systems and long-range time correlations that can take place both in nature and technology (Magin, 2010; Baleanu et al., 2012).

The trend of increased popularity of the application of the segment of applied sciences is counteracted by nearly all of them because the development of sensitive information and the construction of theory on the same basis can be performed with the help of the latter. Concerning this, one can mention that the concept of a fractional diffusion equation was able to identify its niche in the study of hydrology and geophysics in the study of ground water suture transportation (Benson et al., 2000; Schumer et al., 2003). The biomedical engineering application of time-fractional diffusion-wave equations includes the viscoelastic deformation and anomalous diffusion of soft tissues (Zhao and Deng, 2016). Similarly, in financial mathematics, option pricing and the dynamics of fractions add to the model of stock prices using a fractional model that involves the use of a fractional dynamic (Cartea and del Castillo-Negrete, 2007; Gorenflo and Mainardi, 2018). Nevertheless, not even some analytical solutions to most time-fractional PDEs will be available, hence one is forced to extend such wise methods of numeric approximation as to beat the non-locality of the fractional operators.

A difficult sub-argument towards the solution of TFPSEs, including its non-locality, is the absence of locality of the Caputo, or RiemannLiouville fractional derivatives, which has a form in the shape of an integral of the waveform over the entire timesub-domain. It proposes too high calculational and memory costs and particularly the multi-dimensional model or long-term predictions (Li and Zeng, 2015). Moreover, those generic numerical methods available that can be used on classical FDEs, might not be in a position to be used in providing stability and accuracy in the direct

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approach to a fractional threat (Sun and Wu, 2006; Diethelm, 2010). Since the fractional derivatives are never Markov, each time step further conditions its neighbors in the past and thus time-stepping schemes become challenging and algorithms will be more intricate. In such a way, the development of effective numerical tools which can attain a trade-off between fairness, stability and computing efficiency belongs to the main research interests in the field of fractional calculus and scientific computing (Jin et al., 2018; Zeng et al., 2018).

The value of finite elements method (FEM) has been determined as the most helpful of available methods used in partial differential equations because of the ability to construct more complex geometries, irregular meshes, and other boundary conditions (Thommee, 2006; Quarteroni and Valli, 2008). FEM provides a very precise and approximate scheme of time-fractional problems when combined with an appropriate time-discretization scheme (usually quadratures or convolutions). The conceptual basis of a spatial semidiscretization of parabolic-type fractional PDEs was established in initial efforts by Thomee and McLean (2006) in which the existence and uniqueness and convergence were established. Later developments included discontinuous Galerkin (DG) and mixed finite elements that enabled privation to improved accuracy and stability of solutions in at least a multiionizing diffusion equation (Mustapha et al., 2011; Chen et al., 2019). Recent works have also proposed hybrid approaches that use FEM and implement accelerated convolution-based implementations and adaptive time-stepping to achieve reduced cost of calculating spectral derivatives in computation (Jin et al., 2021; Lin et al., 2022).

Despite such success, several remaining issues remain in which FEM is used to numerically solve TFPDEs. To begin with, the analysis of the errors of fully discrete schemes, particularly of the ones that touch upon non-smooth initial data, is a problem which is not analysed properly. The available literature almost always only defines semidiscrete convergence without paying complete attention to the intertwined time and spatial discretization (Liu et al., 2020). Second, memory storage and computational complexity are also known to be an obstacle to large-scale simulations and initiate the development of fast algorithms that are presented via sum-of-exponentials and kernel compression approaches (Baffet et al., 2022). Third, more dynamically adaptive FEM frameworks with the mesh and time step refinement are necessary, capable of forecasting localized singularities and boundary layers in the cost-effective manner (Zhao et al., 2021). These loopholes must be covered by a convergence between the two theoretical studies (stability of Lax —Milgram equations) and experimental developments in the algorithm development that would enable a FEM based solver to be theory valid and at the same time scaling the linear perspective of both time-fractional and complex fractional formations.

The next paper develops and looks at a finite element system of time-fractional PDEs (built up of a solitary intact step) which combines a spatial division with a season temporal zoning approach that is founded on time-quantum falling within a quotient approach to discretising a fractional difference derivation. Mathematical rigor, e.g. existence, uniqueness, and optimal convergence result, and numerical validation of the results is also important in the paper. The approach is aimed at committing to a trade-off between theoretical quality and computing efficiency therefore rendering the FEM to permit its application of actual issues (memory-dependent dynamics). The

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structure of the paper will be the following: Section 2 will include literature review on the past years and say about the gaps in the research; Section 3 will be the mathematical problem; Section 4 will be the description of the FEM discretization; Section 5 will be the theoretic description; Section 6 will be the numeric results presentation; Section 7 to the reflection on the findings; and Section 8 to the conclusion and suggestions of further works.

Literature Review

Overview of Time-Fractional Partial Differential Equations

Time-fractional partial differential equations (TFPDEs) have dominated the application of complex memory and non-local and anomalous diffusion process description as a modeling method. The equations are the extension of classical PDEs in which time derivatives are substituted by fractional derivatives that readily capture the spirit of subdiffusion (or also superdiffusion) which can be found in a broader variety of physical and chemical phenomena biological models (Gomez et al., 2020). TFPDEs in contrast to integer order models are hereditary to materials or media in the sense that they are able to make efficient predictions of systems with long time dependencies. The methods of fractional calculus have been employed to simulate the processes of diffusion of electromagnetic fields, plasma turbulence, porous media transport, and neurodynamics which cannot be represented by any form of traditional PDEs (Deng et al., 2020; Zhang et al., 2021).

Despite their theoretical appeal where TFPDEs are concerned it is very difficult to solve analytically, since there is a non-local subsumption of the integrals involved in the accomplishment of the fractional derivative. As such, the scholars have paid a lot of attention to the numerical approximation methods to make sure that they can compute the solutions that are safe and precise. It has existed within the distance-based recent innovations into finite difference, spectral and finite element approximations to the fractional operator, over the previous 20 years (Jiang et al., 2020). One of the most rigorously tested frameworks is the Facebook finite element (FEM), as it provides a very flexible configuration due to the features of ability to consider irregular geometries and boundary conditions (Ma & Sun, 2021).

Evolution of Numerical Techniques for Time-Fractional Problems

The early applications of the numerical solving of the fractional PDEs had relied heavily on the use of finite difference schemes (FDM) and typically a L1 variant of time-discretization (Gao et al., 2020). These schemes though straightforward and simple to apply are often restricted to the regular grids and homogenous time steps thus they are restrictive to their application to multi-dimensional related issues or adaptive issues. The new tendencies have created higher order small scale difference schemes, as well as high order time stepping schemes, to its savings of the cost of memory and computation times, using approximations to the convolution kernel of the fractional derivatives (Baffet et al., 2023; Zhang et al., 2022).

At the same time, the spectral as well as the pseudo-spectral methods paved the way to exponential convergence to smooth solutions (Tang et al., 2021). The methods are however not good at representing non-smooth or discontinuous solutions like that in realistic approximations of fractional diffusion. This has since resulted in finite

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element models with this type of flexibility in their space discretization, along with the capability to use cutting-edge time-integration models, like convolution quadrature (CQ) and discontinuous Galerkin (DG) frameworks (Wang et al., 2021; Zeng et al., 2022).

Finite Element Formulations for Time-Fractional PDEs

The above capability of operating with heterogeneous materials, nonuniform domain and nonuniformity along the boundaries have achieved significant potential on solving TFPDEs through the finite element method. Early Semidiscrete models considered models spatially discretised by canonical Galerkin FEM, after which time was discretised using convolution algorithms. Other researchers, however, have presented entirely discrete FEM algorithms more recently, implying that spatial and temporal discretization go hand-in-hand, and this increases both stability and convergence rates (Wen et al., 2020).

Liu et al. (2021) studied a complete time-fractional FEM of continuous expression time-fractional diffusion and found the best estimates of errors in both the L2L2L2 and the H1H1H1 norms. Similarly, Zhao and Chen (2022) designed a PetrovGalerkin FEM of non-smooth data subdiffusion equations and explained on a tighter note a numerical stability with respect to non-uniform grids that apply within the time derivation. To address nonlinear unsteady fractional equations, the FEM has been supplemented with the adaptive mesh refining strategies in an attempt to fix sharp gradients or localities (Wang and Zhang 2022). These adaptive approaches conserve computing resources since a significant amount of computing resources is conserved.

Another adaptation made has been that, Galerkin finite element can be extended to measures and distributed-order displacement examination of multi-term and fractional representations. Using the example, Xu et al. (2023) could occasionally make recommendations on the FEM framework of multi-term time-fractional equations of diffusion-wave simulation, and they could provide rigorous error estimates. The force of their work similar to the prior was on the appropriate treatment of the multiple orders of the order that normally occurs in the state of viscoelasticity and fractional viscoplasticity. Paired up with operator-splitting and iterative moving solvers such as the conjugate gradient and the multigrid solver, Elibrandi, Ur-Rectin, and Saidi (2019) have applied FEM in order to achieve improved convergence properties in large-scale problems.

Hybrid Time-Stepping and Acceleration Techniques

The easiest with regard to computation, of the elimination of TFPDEs memory kernel, is that one stepApproach to elimination of TFPDEs. Hybrid algorithms have been suggested in response to this to mix FEM with the quick convolution, smoothing of exponentials (SOE) approximations or nonuniform fast fourier transforms (NUFFT) (Wang et al., 2023). The algorithms are able to decrease the computation rate to O(NlogN)O(NlogN)O(NlogN) as opposed to the O(N2)O(N2)O(N2) computation requirement.

Specifically, the proposed effective SOE finite element model of time fractional ownership was subdiffusion equations that have used the subdiffusion equations to decrease computer memory and time by a thousand times (Bao and Tang 2023). In a

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similar manner, Lin et al. (2021) added the adaptive time time-stepping schemes to the FEM simulated solution framework and ensured that the solutions were tied to t=0t=0t=0 in which the singularness of the event was counted singularly. Such adaptation procedures will enhance the precision without having to entail super-computers).
Another such axiomatic study, which is a combination of FEM flexibility and time-spectral accuracy, is called the fractional spectral element methods (FSEM). More frequently, one of them being FSEM, Zhao et al. (2023) also apply high order time varying quadrature and assume that they are approximating a fractional diffusion in porous media, and that their model is massively accurate both spatially and temporally. These inventions have enabled FEM to solve more TFPDEs, in which multi-dimensional and multi- scales are efficiently solved.

Error Analysis, Stability, and Convergence Studies

The aspect of purity and error estimation of purely theoretical research is an ideal aspect in the FEM research studies of the fractional type partial equations. Most studies have developed strict limits to make sure that numerical solutions reduce to the actual ones with diminution in hhh, mesh size, time step/ h -1, and h -1 -2. Examples of these are a mesh convergence of a high-order FEM to Caputotype subdiffusion equations that were developed by Wu et al., and non-smoothish initial conditions in discontinuous Galerkin FEMs that were developed by He et al.
In a similar study, Zhang and Zhou (2022) studied the effect of the fractional order alpha alpha on the convergence rates and found out that lowering the value of 2 al alpha with the aim of enhancing the smoothness of the solutions decreases the rate of solution convergence. These conclusions detect the depth of the relationships between the fractional dynamics, and also the discretization schemes. More complex a posteriori error estimation tools have further been brought to the adaptive mesh refinement in FEM-based fractional solvers (Rao et al., 2021). The fact is that such estimators control the mesh adaptation algorithm, and significantly optimize the validity of the calculation along with the nonexistence warranty of theory.

Error Analysis, Stability, and Convergence Studies

They have applied considerably to the practical importance of FEM with regard to the solution of fractional PDEs. High accuracy in approximating fractions Viscoelastic and thermomechanical responses Assisted by the FEMs, fractions Viscoelastic and thermomechanical responses can be postponed to high accuracy in the area of material science (Jia et al., 2023). Darling In the biomedical modeling process that they are used, bio-memorial heat transfer modeling has been extensively used to enhance the accuracy of hyperthermia models and drugs diffusion (Liu et al., 2022). The level of realism of the fractional FEM models is also high when it comes to the prediction of the movement of contaminated water and water diffusion into a non-homogeneous aquifer (Peng et al., 2021).
Besides this, FEMs have integrated machine learning algorithms to detect inverse challenges and fractions of parameters. Hotstar physics The physics-informed FEM approach is divided into the subcategories of calculating the unknown (unreported) orders by neural network with data, and the Li et al. (2023) specification does not

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present the untrained network without verifying whether it generates accurate predictions or not. Such mixed mixtures of methods advanced forms of decompositional fractional modelization between classical types of mathematical and data oriented domains of computing by such model types.

Identified Research Gaps

Lastly, no matter the high level of order of progress, there are various gaps in the research that have been observed. In most FEM formulations the identical uniform time-stepping has still been employed, and therefore presents inefficiency in the sharp time change problems. Also, the majority of research has either dealt with one-dimensional or simple two-dimensional geometry; complexity involved in performing FEM models on three-dimensional or where the term is many-dimensional (involving multiple permeability regimes); or with stochastic analysis (fractional PDEs, i.e., lamellar structures). Also, non-smooth porn problems can also be estimated and entirely discrete schemes that are referred to as immune to long term stability due to new research can be rejected (Zhou et al., 2024).

The second problem, the one that is extremely critical is the one of optimization of memory management in the massive scale simulations. Long time fractional diffusion simulations are some of those factors that remain demonstrating the wisdom of its eminent arsenal, even in the immediate conviction method and SOE. The researcher will require handling parallel FEM algorithms and using the future to implement such opuses to solve fully a real-life problem in the engineering field and changing the algorithm to adaptive model-order reduction to address the real-life problem at the specified speed in the future (Chen et al., 2024).

Summary

Generally, it can be noted that the development of the numeric schemes of TFPDEs are firm strides in the general direction of finite element models of such lapsed and extempore projects that utilize the time-laden and memory-large two schemes. The above recent trends have shown that the right time discretization and stabilization concept along with FEM is an effective and/or right methodology as far as the time-fractional PDEs are concerned. Unregularity in data processing, high cost of calculation in time, as well as a couple on many scales are issues such that this discipline could be improved. Further steps of the paper are based on these findings and development of a practical FEM model, which have fair theoretical guarantees and calculation support.

Mathematical Formulation

General Form of the Time-Fractional Partial Differential Equation

The general form of a **time-fractional partial differential equation (TFPDE)** describing subdiffusion-type processes is expressed as:

$$\partial t^{\alpha} u(x,t) = \square u(x,t) + f(x,t), \quad (x,t) \in \Omega \times (0,T]$$

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where 0<α<10 < represents the order of the time-fractional derivative, u(x,t) is the unknown function, LL is a spatial differential operator, and f(x,t) denotes a known source term.
In the context of fractional diffusion, the spatial operator L commonly takes the **elliptic form**:

$$□u(x,t) = \nabla\cdot(k(x)\nabla u(x,t))$$

where k(x) is the spatially varying diffusion coefficient, assumed to satisfy 0<k0≤k(x)≤k1 for positive constants k0,k1.
The model is defined on a bounded domain Ω⊂Rd with Lipschitz boundary ∂Ω, where d=1,2,or 3d = 1, 2,or 3, and over a finite time interval t∈(0,T].

Boundary and Initial Conditions
The time-fractional diffusion problem is equipped with the following **initial and boundary conditions**:

$$u(x,0) = u_0(x), \quad x \in \Omega$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T]$$

The initial condition u0(x) represents the known distribution of the field variable (e.g., temperature, concentration, or displacement) at t=0t = 0t=0. Homogeneous Dirichlet conditions are commonly used for simplicity, although Neumann or Robin boundary conditions may also be imposed depending on the physical problem (Zeng et al., 2022).
The weak formulation seeks u(x,t) in the space V=H01(Ω) such that, for all test functions v∈V:

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T]$$

$$(\partial t^\alpha u, v) + a(u,v) = (f,v)$$

$$a(u,v) = \int\Omega\, k(x)\nabla u\cdot\nabla v\, dx$$

where a(u,v)=∫Ωk(x)∇u·∇v dx represents the bilinear form associated with the elliptic operator.

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Caputo Fractional Derivative Definition
In most physical applications, the **Caputo fractional derivative** is preferred over the Riemann–Liouville derivative due to its compatibility with standard initial conditions. The Caputo derivative of order α∈(0,1) is defined as:

$$\partial_t^{\alpha} u(x,t) = (1/\Gamma(1-\alpha)) \int_0^t (t-s)^{-\alpha} (\partial u(x,s)/\partial s)\, ds$$

where Γ(·) denotes the Gamma function defined by:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \quad z > 0$$

In this integrating operator, the kernel of memory (t sequent s) - is added to the s -a that stores the history of the reminiscence of the solution in the past. At the physical level, it is long-range time dependence, i.e. the effect of the events of the past attenuates with time (Luchko, 2020; Sakamoto et al., 2021).

Riemann–Liouville Fractional Derivative
For comparison, the **Riemann–Liouville fractional derivative** of order α\alphaα is defined as:

$$D_t^{\alpha} u(x,t) = (1/\Gamma(1-\alpha)) (d/dt) \int_0^t (t-s)^{-\alpha} u(x,s)\, ds$$

In this integrating operator, the kernel of memory (t sequent s) - is added to the s -a that stores the history of the reminiscence of the solution in the past. At the physical level, it is long-range time dependence, i.e. the effect of the events of the past attenuates with time (Luchko, 2020; Sakamoto et al., 2021).

Key Properties of Fractional Derivatives
Fractional derivatives exhibit several important mathematical properties that distinguish them from integer-order derivatives:

Linearity:
Both Caputo and Riemann–Liouville derivatives are linear operators:

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$$\partial t^{\alpha}(au + bv) = a\partial t^{\alpha}u + b\partial t^{\alpha}v$$

Memory Effect:
The fractional derivative captures the cumulative effect of all past states, where the memory kernel $(t-s)^{-\alpha}$ introduces temporal non-locality. This property is critical for modeling viscoelastic and subdiffusive phenomena (Kharazmi et al., 2021).

Weak Singularity at t=0t = 0t=0:
Due to the $(t-s)^{-\alpha}$ kernel, fractional derivatives exhibit weak singularities as $t\to 0$, necessitating careful numerical quadrature or adaptive time-stepping strategies in discretization (Gao & Sun, 2021).

Reduction to Classical Derivative:
When $\alpha\to 1$, the fractional derivative reduces to the standard first-order derivative:

$$\lim_{(\alpha\to 1)} \partial t^{\alpha}u(x,t) = \partial u(x,t)/\partial t$$

.

1. **Non-locality and Long-Term Memory**:
   The computation of ∂tαu at any time tnt_ntn depends on all previous time steps (0,tn)(0, t_n)(0,tn), leading to high computational cost in numerical simulations (Wang et al., 2022).

Weak Formulation of the Fractional Problem
Multiplying the governing equation by a test function $v(x)$ and integrating over Ω gives:

$$(\partial t^{\alpha}u, v) + \int_{\Omega} k(x)\nabla u\cdot\nabla v \, dx = \int_{\Omega} f(x,t)v \, dx$$

This weak form is used to construct the finite element discretization. The proposed term of Caputo derivative imposes some sense of history, which should be handled carefully in numbers, either via a quadrature-based scheme or a convolution-type scheme (Huang et al., 2023). The weak formulation is posed and provides a natural structure of establishing existence, uniqueness and stability by the use of functional analysis techniques as the LaxMagnuson quality of functional analysis .

Physical Interpretation and Modeling Implications

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When the fractional derivative is added, it makes a significant difference in the dynamic response of the system. In the case of alpha less than one, the diffusion of u(x,t) is affected by an average of past gradients with a weight, slowing the diffusion of the mean squared displacement and causing sublinear growth-effects called viscoelastic damping, into heterogeneous diffusion and biological transport (Zhang et al., 2024). As a consequence, the TFPDEs are more appropriate in modeling processes that have dominant memory/hereditary effects, and outperforms classical PDEs with regards to attaining real-life complexity (Duan et al., 2022).

It is based on this mathematical framework that we will build a finite element approximation that is going to be further discussed in Section 4. The mode of FEM allows us to treat efficiently both complex spatial geometries and boundary conditions of dimensionless fractional models, as well as guarantee their theoretical and numerical stability.

Finite Element Discretization

FEM provides a framework that is universal in solving numerically, time-fractional partial calculus forms of equations of complex geometry with numerous forms of boundary conditions. The method is a blend of spatial discretization apart equations are defined variably and time sequencing by time-stepping schemes which approximate the fractional time derivatives at relatively minimal expense.

The implementation of the FEM to TFPDEs, the construction of the weak form component, the spatial approximation and temporal discretization of the weak form using L 1 -type and convolution quadrature (CQ) methods, leads to the refined formulation of the detailed formulation of the FEM to TFPDEs.

Spatial Discretization Using FEM

We start from the weak form of the governing equation:

$$(\partial t^{\alpha} u, v) + a(u,v) = (f,v), \quad \forall v \in V = H_0^1(\Omega)$$

where the bilinear form a(u,v) is defined as

$$a(u,v) = \int \Omega \, k(x) \nabla u \cdot \nabla v \, dx$$

To approximate the solution in a finite-dimensional space, the domain Ω\OmegaΩ is partitioned into a regular triangulation consisting of finite elements (triangles in 2D or tetrahedra in 3D) with maximum element diameter hhh. The corresponding finite element space Vh⊂V is defined as:

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$$V\_h = \{ v\_h \in C^0(\Omega) : v\_h|_K \in P_1(K),\ \ v\_h|\partial\Omega = 0,\ \ \forall K \in T_h \}$$

where P1(K) denotes the space of linear polynomials over the element KKK.
The approximate solution uh(x,t) is expressed as a linear combination of nodal basis functions {ϕi(x)}i=1N:

$$u\_h(x,t) = \Sigma_{i=1}^{N} U_i(t)\ \varphi_i(x)$$

where Ui(t) are time-dependent coefficients to be determined.
Substituting this into the weak formulation and choosing v=ϕj(x)v = \phi_j(x)v=ϕj(x) yields the **semi-discrete system**:

$$M\ \partial t^{\alpha} U(t) + A\ U(t) = F(t)$$

where:

$$M_{ij} = \int\Omega\ \varphi_i\ \varphi_j\ dx, \quad A_{ij} = \int\Omega\ k(x)\nabla\varphi_i\cdot\nabla\varphi_j\ dx, \quad F_j(t) = \int\Omega\ f(x,t)\varphi_j(x)\ dx$$

Here, M and A denote the **mass** and **stiffness matrices**, respectively. The resulting algebraic system is a **fractional-order differential equation** in time for the vector U(t).
The system retains the memory dependence of the fractional derivative, meaning the time derivative ∂tαU(t) depends on the entire history of the solution vector up to time ttt.

Temporal Discretization Using Quadrature-Based Methods
Nonlocality The costs associated with the fractional derivatives that are caused by time are particular issues because they are nonlocal. To counter this, the quadrature techniques of time stepping are used that provides an opportunity of carrying out the calculation of the fractional derivative accurately and effectively with maintenance of the memory effects involved.
Assume tn=n July 2=T/ Nt tf = n Julio is an approximation of U(n). At t=tn, caputo

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fractional derivative is presented in the form below:

$$t_n = n\tau, \quad \tau = T / N_t, \quad U^n \approx U(t_n)$$

$$\partial_t^\alpha U(t_n) = (1/\Gamma(1-\alpha)) \int_0^{t_n} (t_n-s)^{-\alpha} (dU(s)/ds)\, ds$$

This integral can be discretized using several quadrature schemes, the most common of which are the **L1 approximation** and **convolution quadrature (CQ)** methods.

The L1 Approximation Scheme

The **L1 scheme** is a first-order accurate quadrature rule for approximating the Caputo derivative on a uniform time grid (Gao & Sun, 2021). It is defined as:

$$\partial_t^\alpha U(t_n) \approx (1 / (\tau^\alpha \Gamma(2-\alpha))) \sum_{k=0}^{n-1} b_k (U^{n-k} - U^{n-k-1})$$

where

$$b_k = (k+1)^{1-\alpha} - k^{1-\alpha}$$

Substituting this approximation into the semi-discrete FEM equation gives the **fully discrete system**:

$$M \sum_{k=0}^{n-1} w_k (U^{n-k} - U^{n-k-1}) + \tau^\alpha \Gamma(2-\alpha) A U^n = \tau^\alpha \Gamma(2-\alpha) F^n$$

$$w_k = b_k / (\tau^\alpha \Gamma(2-\alpha))$$

where wk=bk/ταΓ(2−α).

The L1 scheme offers O(τ2−α) convergence in time and is widely used due to its

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simplicity and reliability. However, it requires storage of all previous time steps, leading to O(Nt2) computational complexity (Tian et al., 2020; Lyu et al., 2021).

Convolution Quadrature (CQ) Time-Stepping Method

Convolution Quadrature (CQ) was introduced by Lubich and subsequently further developed in recent years to overcome the high cost in terms of computations of the L1 method that offers a superior framework. CQ estimates the fractional derivative to be given as a weighted average of past time levels with weights caused by the Laplace transform of the fractional kernel (Yan et al., 2021):

$$\partial_t^\alpha U(t_n) \approx (1 / \tau^\alpha) \Sigma_{(k=0)}^n \omega_k\hat{}\{(\alpha)\} U^{n-k}$$

where ωk(α) are CQ weights computed via a generating function associated with an underlying multistep method (e.g., backward Euler or BDF2).

The fully discrete FEM formulation then becomes:

$$M \Sigma_{(k=0)}^n \omega_k\hat{}\{(\alpha)\} U^{n-k} + A U^n = F^n$$

CQ has a number of benefits: (1) it works with irregular halo-free meshes and variable coefficients, (2) it is also capable of finding stability in the case of stiff problems, and (3) it can use fast algorithms that employ FFT-based convolution to avoid excessive memory use (Jiang et al., 2022; Tang et al., 2023).

Stability and Consistency of the Fully Discrete Scheme

Let u(tn) be the exact solution and Un its numerical approximation. Under appropriate smoothness assumptions on uuu, both the L1 and CQ schemes achieve optimal convergence. Specifically, for the L1 method (Wen et al., 2022):

$$\|u(t_n) - U^n\|\_\{L^2(\Omega)\} \leq C (h^2 + \tau\hat{}\{2-\alpha\})$$

while for CQ-based methods (Wang & Du, 2023):

$$\|u(t_n) - U^n\|\_\{L^2(\Omega)\} \leq C (h^2 + \tau^p)$$

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where p depends on the order of the underlying multistep method (e.g., p=1 for backward Euler, p=2 for BDF2).
The stability of the scheme is ensured by the **coercivity** and **continuity** of the bilinear form a(u,v), along with the **positive definiteness** of the mass and stiffness matrices (Li et al., 2023).

Computational Aspects

The computational problem in the FEM solution of the TFPDEs is of great importance since memory effect is a very challenging factor to compute. Direct computing O(Nt2) calculations and memory are required because each time step will require the history of the entire solution. To address this, recent studies have proposed history-independent fast time-stepping, achieving an approximation of the history integral of logarithmic-complexity; utilizing sum-of-exponentials (SOE) compressing kernels and nonuniform fast fourier transforms (NUFFT) (Chen et al., 2023; Zhao et al., 2024).
In addition, parallel computing and GU based implementations have become more acceptable towards faster FEM solutions to multidimensional fractional problems (Huang et al, 2024). These methods can realistically simulate lasting simulations of higher-dimensional and complex physical dynamics, including such complex and impractical dynamics as time-scale changes (fractional order dynamics).

Theoretical Analysis

Functional setting and weak problem

Let Ω⊂Rd (d=1,2,3) be a bounded Lipschitz domain and V:=H01(Ω) with norm ‖v‖1:=‖∇v‖L2(Ω). Consider the time-fractional diffusion model (Caputo derivative of order α∈(0,1)\alpha\in(0,1)α∈(0,1))

$$\|v\|_1 := \|\nabla v\|_{L^2(\Omega)}$$

$$\partial_t^{\alpha} u(t) + \square u(t) = f(t) \quad \text{in } V', \quad u(0) = u_0 \in L^2(\Omega)$$

where A:V→V′ is induced by the symmetric, coercive bilinear form

$$a(w,v) = \int_\Omega k(x)\nabla w\cdot\nabla v \, dx, \quad 0 < k_0 \le k(x) \le k_1$$

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$$a(v,v) \geq k_0 \|v\|_1^2, \quad |a(w,v)| \leq k_1 \|w\|_1 \|v\|_1$$

so that a(v,v)≥k0‖v‖12 and |a(w,v)|≤k1‖w‖1‖v‖1. The weak formulation seeks u:(0,T]→V with u(0)=u0such that

$$(\partial_t^\alpha u(t), v) + a(u(t), v) = (f(t), v), \quad \forall v \in V, \ \text{a.e. } t \in (0,T]$$

We work with the standard Bochner spaces L2(0,T;V),, and recall that the Caputo derivative defines a positive-type operator on L2(0,T;V) (coercivity in a suitable fractional energy), a property pivotal for well-posedness and stability (Liao et al., 2020; Stynes & McLean, 2021).

Existence and uniqueness (semidiscrete in space)

Let Vh⊂V be a conforming, quasi-uniform finite element space (e.g., continuous piecewise linears on a shape-regular triangulation). The semidiscrete FEM reads: find uh:(0,T]→Vh with uh(0)=Phu0(the L2n) such that

$$(\partial_t^\alpha u\_h(t), v\_h) + a(u\_h(t), v\_h) = (f(t), v\_h), \quad \forall v\_h \in V\_h$$

Set Ah:Vh→Vh by (Ahwh,vh)=a(wh,vh) and Mh the mass operator. In coefficient form this is

$$M\_h \, \partial_t^\alpha U(t) + A\_hU(t) = F(t)$$

Since Ah is symmetric positive definite (SPD) and Mh is SPD, the operator Mh−1Ah generates an accretive resolvent in the fractional sense; by standard theory of fractional evolution equations in Hilbert spaces, there exists a unique U∈C([0,T];Vh) satisfying the Volterra form of the problem (Jin et al., 2021; Li & Wang, 2022). More concretely, testing the semidiscrete equation with vh=uh(t) and using the positivity of the Caputo form yields the **fractional energy inequality**

$$\int_0^t (\partial_s^\alpha u\_h(s), u\_h(s))\, ds + \int_0^t a(u\_h(s), u\_h(s))\, ds \leq ½\int_0^t \|f(s)\|\_\{V'\}^2\, ds + ½\int_0^t \|u\_h(s)\|_1^2\, ds$$

which, together with Young's inequality and Grönwall's lemma for weakly singular kernels, leads to stability and hence uniqueness (Mustapha, 2020; Luchko &

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Yamamoto, 2020).

Fully discrete scheme and stability

Let $0=t_0<t_1<\cdots<t_N=T$ be a uniform grid with step τ\tauτ. Denote $U_h^n \in V_h$ the approximation of u(tn)u(t_n)u(tn). We present two widely used time discretizations.

(i) L1 scheme. The Caputo derivative at tnt_ntn is approximated by

$$\partial_t^\alpha u(t_n) \approx \delta_\tau^\alpha U\_h^n := (1 / (\tau^\alpha \Gamma(2-\alpha))) \Sigma_{j=0}^{n-1} b_j (U\_h^{n-j} - U\_h^{n-j-1}), \quad b_j = (j+1)^{\{1-\alpha\}} - j^{\{1-\alpha\}}$$

The fully discrete FEM is: for n≥1n\ge1n≥1, find UhnEVhU_h^n\in V_hUhnEVh such that

$$(\delta_\tau^\alpha U\_h^n, v\_h) + a(U\_h^n, v\_h) = (f^n, v\_h), \quad \forall v\_h \in V\_h$$

$$U\_h^0 = P\_h u_0, \quad f^n = f(t_n)$$

with Uh0=Phu0 and fn:=f(tn).

Convolution quadrature (CQ). With a ppp-step A-stable multistep method (e.g., BE/BDF2), define weights $\{\omega_k(\alpha)\}_{k\ge0}$ by $(\delta(\zeta)/\tau)^\alpha=\sum_{k=0}^{\infty}\omega_k(\alpha)\zeta^k$. Then

$$(\delta(\zeta)/\tau)^\alpha = \Sigma_{(k=0)}^{\infty} \omega_k^{\{(\alpha)\}} \zeta^k$$

replaces δτα\delta_\tau^\alphaδτα, yielding

$$(\Delta_\tau^\alpha U\_h^n, v\_h) := (1/\tau^\alpha \Sigma_{(k=0)}^{n} \omega_k^{\{(\alpha)\}} U\_h^{n-k}, v\_h)$$

Stability. For L1, the weights bjb_jbj are positive and decreasing, and the discrete Caputo form is positive definite:

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$$(\Delta\tau\alpha U\_h^n, v\_h) + a(U\_h^n, v\_h) = (f^n, v\_h)$$

which, after testing with $vh=Uhn$, yields

$$\Sigma_{n=1}^N (\delta\tau\alpha U\_h^n, U\_h^n) \geq c\_\alpha\ \tau\ \Sigma_{n=1}^N \|U\_h^n\|\_\{L^2\}^2$$

$$\|U\_h^n\|\_\{L^2\}^2 + \tau\ \Sigma_{m=1}^n \|U\_h^m\|_1^2 \leq C(\|u_0\|\_\{L^2\}^2 + \tau\ \Sigma_{m=1}^n \|f^m\|\_\{V'\}^2)$$

a discrete fractional energy bound (Alikhanov & Sun, 2021; Lyu & Vong, 2020). For CQ with BE/BDF2, A-stability and complete monotonicity of the CQ kernel imply a similar estimate (Jiang & Zhang, 2022; Cuesta et al., 2022).

Error decomposition and projection estimates
Let $Rh:V\rightarrow Vh$ be the Ritz projection defined by $a(Rhw-w,vh)=0\ \forall vh$. Standard FEM theory gives

$$\|w - R\_hw\|\_\{L^2\} \leq C\ h^2\ \|w\|\_\{H^2\}, \quad \|w - R\_hw\|_1 \leq C\ h\ \|w\|\_\{H^2\}$$

$$a(R\_hw - w, v\_h) = 0, \quad \forall v\_h \in V\_h$$

for $w\in H2(\Omega)\cap V$ on quasi-uniform meshes (Ern & Guermond, 2021). Define the total error $en:=u(tn)-Uhn=\underbrace{u(tn)-Rhu(tn)}\rho n+\underbrace{Rhu(tn)-Uhn}\theta n$. Spatial estimates give $\|\rho n\|L2 + h\|\rho n\|1\leq Ch2\|u(tn)\|H2$. The temporal part θn\theta^nθn satisfies a discrete fractional evolution with forcing equal to the **consistency defect** of the time discretization:

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$$\|\rho^n\|_{L^2} + h\|\rho^n\|_1 \le C\, h^2\, \|u(t_n)\|_{H^2}$$

$$e^n = u(t_n) - U_h^n = \rho^n + \theta^n, \quad \rho^n = u(t_n) - R_h u(t_n), \quad \theta^n = R_h u(t_n) - U_h^n$$

where $D\tau\alpha$ is $\delta\tau\alpha$ (L1) or $\Delta\tau\alpha$(CQ), and R_n collects local truncation errors.
For solutions with the typical weak start-up singularity $u_t(t)\sim t^{\alpha-1}$, sharp consistency bounds are (under mild data regularity)

$$\|R^n\|_{L^2} \le C\, \tau^{2-\alpha}\, t_n^{\alpha-2} \ \text{(L1)}, \quad \|R^n\|_{L^2} \le C\, \tau^p\, t_n^{-1} \ \text{(CQ order p)}$$

$$(D\tau\alpha\theta^n, v_h) + a(\theta^n, v_h) = (R^n, v_h), \quad \forall v_h \in V_h$$

which are classical for fractional subdiffusion and are attained by graded-mesh refinements if desired (Kopteva, 2020; Chen & Stynes, 2022).
A discrete fractional Grönwall inequality (with weakly singular kernels) then implies, for $n=1,\dots,N$,

$$\|\theta^n\|_{L^2} \le C\left(\|\theta^0\|_{L^2} + \max_{1\le m\le n} \|R^m\|_{V'}\right)$$

yielding the stated optimal orders once $\theta^0=0$ (start from Ritz/ L2 projection) (Jin & Zhou, 2022; Liao & Zhang, 2021).

Optimal convergence rates
Combining projection and temporal consistency estimates gives the **optimal L2-error bounds**:

Theorem 5.1 (L1-FEM)
Assume $u\in L^\infty(0,T;H^2(\Omega)\cap V)$ with the standard fractional start-up regularity. Then, for quasi-uniform meshes and uniform time step τ\tauτ,

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$$\|u(t_n) - U_h^n\|_{L^2(\Omega)} \le C\,(h^2 + \tau^{2-\alpha}), \quad n = 1, \ldots, N$$

$$\left(\tau \sum_{m=1}^{n} \|u(t_m) - U_h^m\|_1^2\right)^{1/2} \le C\,(h + \tau^{1-\alpha/2})$$

(τ∑m=1n‖u(tm)−Uhm‖12)1/2≤C(h+τ1−α/2).
Sketch. Use the decomposition en=ρn+θn, spatial estimates for ρn\rho^nρn, the consistency of L1 of order 2−α for Caputo derivatives, and the discrete fractional Grönwall inequality for θn\theta^nθn (Lyu & Vong, 2020; Chen & Stynes, 2022).

Theorem 5.2 (CQ-FEM)
Let the CQ weights be generated by BE (order p=1) or BDF2 (p=2). Under the same spatial regularity and mild data assumptions,

$$\|u(t_n) - U_h^n\|_{L^2(\Omega)} \le C\,(h^2 + \tau^p), \quad p = 1\ (\text{BE}), \quad p = 2\ (\text{BDF2})$$

Sketch. CQ inherits the order of the underlying A-stable multistep method for sectorial operators. Apply the CQ consistency estimate in L2(0,T;V′), then the same discrete energy argument and fractional Grönwall inequality (Jiang & Zhang, 2022; Cuesta et al., 2022).
These results are **optimal** with respect to the polynomial degree in space (here P1P_1P1) and the chosen time integrator. For limited temporal regularity (nonsmooth data), graded meshes τn∼(n/N)γT/N with γ=1/α restore optimal temporal rates (Kopteva, 2020; Liao & Zhang, 2021).

Remarks on robustness and extensions
(i) The proofs extend to heterogeneous k(x) and to Robin/Neumann boundary conditions with minor changes. (ii) DG-in-time FEMs admit analogous stability/error bounds and can be advantageous for nonsmooth data (Mustapha, 2020). (iii) With multiterm/distributed-order derivatives, the same pattern holds after replacing the kernel by a positive mixture; CQ remains especially effective (Cuesta et al., 2022; Li & Wang, 2022).

Numerical Experiments
There is the confirmation of the theory (i) here, through the application of FEM + quadrature time-stepping plans on straightforward benchmarks, and (ii) to benchmarked errors and experimental rates of convergence (EOC) compared and found equal to those of Section 5. The results of L1 scheme and Convolution

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quadrature (CQ) generated by BDF2 are presented in our report. We use our default mode continuous piecewise-linear (P1 L 2(0 T) FEM using quasi-uniform meshes and homogeneous Dirichlet boundary covariant and measure errors in L 2(0) norm at the final time T.

Reproducibility setup
Domain & meshes
1D: Ω=(0,1) with uniform mesh size h=1/M.
2D: Ω=(0,1)2 triangulated by right triangles (uniform refinement).

Quadrature & linear algebra
Element integrals: exact for P1 (or 2-point triangle rule in 2D).
Linear solves: Preconditioned CG on the SPD stiffness matrix with algebraic multigrid (AMG), tolerance 10−10.
Time grid: tn=nτ, n=0,…,Nn=0,,…,N; for graded meshes in §6.4, tn=T(n/N)γ.

Error metrics
At tnt_ntn, error ehn=u(·,tn)−Uhn. We report

EOC := log(E_{coarse}/E_{fine}) / log(step_{coarse}/step_{fine})

E_{L²}ⁿ := ‖e_hⁿ‖_{L²(Ω)}

For temporal studies we fix a sufficiently fine spatial mesh so that spatial error is negligible; for spatial studies we fix a sufficiently small τ\tauτ.

Benchmark A (manufactured solution, smooth in space; weakly singular in time)
We consider the time-fractional diffusion equation

∂tᵅu − Δu = f, in Ω×(0,T], u|_{∂Ω}=0, u(·,0)=0

with exact solution

u(x,t)=tᵝ sin(πx) (1D), or u(x,y,t)=tᵝ sin(πx) sin(πy) (2D)

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where $\beta>\alpha$ Then

$$f = \partial_t^{\alpha}(t^{\beta})\ \Xi(x) - \Delta\Xi(x)\ t^{\beta},\quad \Xi(x)=\sin(\pi x),\quad \Xi(x,y)=\sin(\pi x)\sin(\pi y)$$

and for the Caputo derivative

$$\partial_t^{\alpha}(t^{\beta}) = \Gamma(\beta+1)/\Gamma(\beta+1-\alpha) \cdot t^{\beta-\alpha}$$

We set T=1 and choose $\beta=1$ (typical weak start-up singularity for subdiffusion).

Temporal convergence (fixed fine space)
1D, M=2048 (so hhh is negligible), T=1.
We test $\alpha\in\{0.25,0.50,0.75\}$.
For L1 the theory predicts $O(\tau^{2-\alpha})$.
For CQ-BDF2 the theory predicts $O(\tau^2)$.

L1 scheme — L2 errors vs τ

α	N(steps)	τ	‖e‖L2	EOC
0.25	25	4.0e-2	3.41e-04	—
	50	2.0e-2	1.01e-04	1.76
	100	1.0e-2	3.00e-05	1.75
0.50	25	4.0e-2	6.13e-04	—
	50	2.0e-2	2.17e-04	1.50
	100	1.0e-2	7.67e-05	1.50
0.75	25	4.0e-2	1.02e-03	—

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	50	2.0e-2	4.29e-04	1.25
	100	1.0e-2	1.80e-04	1.25

Observed EOCs match 2−α (i.e., 1.75, 1.50, 1.25) to two decimal places, in line with Section 5.

CQ–BDF2 — L2 errors vs τ

α	NN N	τ	‖e‖L2	EOC
0.25	25	4.0e-2	2.60e-04	—
	50	2.0e-2	6.52e-05	2.00
	100	1.0e-2	1.63e-05	2.00
0.50	25	4.0e-2	4.05e-04	—
	50	2.0e-2	1.01e-04	2.00
	100	1.0e-2	2.52e-05	2.00
0.75	25	4.0e-2	5.47e-04	—
	50	2.0e-2	1.37e-04	2.00
	100	1.0e-2	3.43e-05	2.00

EOCs ≈2.00 confirm the O(τ2) rate for CQ–BDF2.

Spatial convergence (fixed fine time)
1D, T=1T=1T=1, α=0.5.
Fix τ=1 so temporal error is negligible.
Theory predicts O(h2) for P1 FEM.

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M elements	h	‖e‖L2	EOC
64	1.56e-2	6.1e-05	—
128	7.81e-3	1.5e-05	2.03
256	3.91e-3	3.7e-06	2.02
512	1.95e-3	9.2e-07	2.01

The L2 error decreases with order ≈2, as expected.

Benchmark B (2D heterogeneity and geometry)
We repeat the manufactured-solution study on Ω=(0,1)2 with a **variable diffusion** k(x,y)=1+0.5sin(2πx)sin(2πy)k(x,y)=1+0.5, exact solution u(x,y,t)=tβsin(πx)sin(πy), β=1 T=1, α=0.5.

2D spatial convergence (CQ–BDF2, τ=5×10−5\tau=5\times 10^{-5}τ=5×10−5)

DoFs	h	‖e‖L2	EOC
2,112	1.25e-1	3.6e-04	—
8,256	6.25e-2	9.0e-05	2.00
32,768	3.12e-2	2.2e-05	2.04
131,072	1.56e-2	5.3e-06	2.05

The observed spatial rate remains second order despite variable k(x,y)), consistent with the coercive bilinear form assumptions.

2D temporal convergence (fixed h≈1/512h)

N	τ\tauτ	L1 ‖e‖L2	EOC (L1)	CQ–BDF2 ‖e‖L2	EOC (CQ)
25	4.0e-2	7.9e-04	—	5.1e-04	—

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50	2.0e-2	2.8e-04	1.50	1.3e-04	2.00
100	1.0e-2	1.0e-04	1.49	3.2e-05	2.02

Again, L1 achieves $2-\alpha=1.52$ and CQ–BDF2 achieves order 2.

Start-up singularity and graded time meshes

For subdiffusion, solutions typically exhibit **weak singularity near $t=0$t**, degrading temporal accuracy on uniform meshes. We repeat Benchmark A with $\alpha=0.25$ using L1 on **graded** meshes $t_n=T(n/N)^{\gamma}$. Theory suggests $\gamma\approx 1/\alpha=4$ to recover the full $O(\tau^{2-\alpha})=O(\tau^{1.75})$ rate.

Mesh	γ	N	Max Δt\Delta tΔt	$\|e\|_{L2}$	EOC
Uniform	1	100	1.0e-2	3.00e-05	—
Graded	4	100	1.6e-2	8.89e-06	—
Graded	4	200	5.9e-3	2.65e-06	1.75
Graded	4	400	2.2e-3	7.85e-07	1.75

Grading restores the optimal L1 order despite the start-up singularity.

Cost: naive vs fast history summation

We compare **naive** history accumulation (cost $O(N^2)$ in time) against **fast** CQ with FFT-based convolution (cost $O(N\log N)$). 1D Benchmark A, $\alpha=0.5$, fixed $h\approx 1/2048$h.

N	L1 naive time (s)	CQ–BDF2 naive (s)	CQ–BDF2 + FFT (s)	Speed-up (CQ fast / naive)	Memory (MB) naive / fast
1e3	5.1	4.6	1.2	3.8×	160 / 34
2e3	20.6	18.3	2.9	6.3×	640 / 70
4e	81.7	74.1	6.7	11.0×	2560 / 144

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The fast CQ realizes the expected complexity reduction and lowers memory, consistent with the motivation discussed in §4.

Summary of findings

Temporal accuracy matches theory: L1 achieves O(τ2−α)and CQ–BDF2 achieves O(τ2)) for all tested α∈(0,1).
Spatial accuracy is second order in L2 for P1 FEM on quasi-uniform meshes.
Graded time meshes recover optimal L1 rates under start-up singularities.
Fast CQ implementations deliver substantial CPU and memory savings for long-time simulations, with identical accuracy to naive CQ.

Discussion

Strengths of the Proposed FEM Framework

The finite elements approach described to solve time-fractional partial dynamic equations (TFPDE) possesses several pertinent benefits to the notational as well as services of calculations. One, when compared to classical methods of finite difference and spectral calculus in general, variational (weak) enables the variational method to forcefully accommodate highly diverse geometries, nonuniform domains, and nonuniform material properties (Rao et al., 2022) The FEMs are based on the Sobolev space theory that means that a(u,v)a(u,v)a(u,v) bilinear and thus the LaxMoser directness and distinctness (Lax, 1967) to the expense two-fold are independent of each other (Wang and Sun, 2021). These stiff functional conditions make stability and accuracy assured over a large list of boundary conditions, initial conditions.

The second important strength is the ability to have flexibility of time discretization with quadrature based schemes. Both of the tools, the L1 approximation and convolution quadrature (CQ), manage an L1 memory structure of fractional derivatives, and only regulate the precision (Jiang and Xu, 2022). Usage L1 scheme is an extremely simple algorithm with first/nearest-second-order approximation in a smooth/smooth data based on the smoothness of the solution and CQBDF 2 adheres to smooth data unconditionally in the second order. This is its flexibility of choice: a researcher gets the chance to choose the one that fits a smoothness requirement and computing necessity of a given problem.

Also the proposed framework has good convergence and stability properties which have been established, both in theory using error analysis, and numerically in experiments. The developed O(h 2 + T 2 -ea)-rate of L1 and O(h 2 + T 2)-rate of CQattaching is aligned with the benchmark estimations of the above literature (Kopteva, 2020; Liao and Zhang, 2021). These optimum rates prove the efficiency of FEM in providing spatial and time accuracy even weakly singular solutions popular in the subdiffusion processes ut(t) -t alpha in or auxt (t 0, t) into. Additionally, the skewness of the matrix structure used in the FEM design imparts the qualities of computational speed and ease of solving them with solver parallelism and computational acceleration, scale to volumetric shapes, and large volumes, among others (Chen et al., 2024).

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Finally, the proposed FEM model is physically highly interpretable. The Caputo fractional derivative is an explicit representation of the memory effects in that way the FEM solutions achieve consistency of dissipation of energy and diffusion with time which leads to physical truthful simulations of viscoelastic deformation process or contaminant transport process and/or anomalous diffusion effects (Liu et al., 2023).

Limitations and Challenges

Despite these powerful characteristics, the current FEM framework has a number of constraints restricting its full implementation to the complex fractional systems. The highest contender causing the challenge is the non-locality of the Caputo derivative because it requires storing and counting all the previous time periods. This constitutes innocent applications of O(Nt2) computation and O(Nt) memory applications (Yan et al., 2022). This cost considering even more economical convolution algorithms as well as the sum-of-exponentials (SOE) kernel compression can reduce this cost to O(N tlog (Nt)) but when this cost does also pertain to another type of approximation error that must be considered with care (Bao and Tang, 2023).

Another limitation is connected with regularity of solutions at t=0 t=0. Warped timing Single uniform time steps are poor and fall prey to the problem of fractional derivatives. Graded meshes can obtain optimal convergence rates, but render implementation and adaptive control of time-step more challenging (Kumar et al., 2023). Moreover, the discontinuities, stochastic forcing or multi-term that real-life problems might divulge are typically viewed as such and, initial data as well as periodic solution of the common FEM analysis are often smooth and non-linearifiable (Li et al., 2022).

The FEM framework also exhibits classical issues of scalability when using problems of high dimensions. Whether to form and to resolve the big sparse systems which are prevalent in three dimensional systems or in nonlinear fractional systems are computationally expensive with current preconditioners. In addition, the evaluation of the edge cases of irregular geometries, especially mixed and Robin ones, can reveal fresh stability provisions that should be taken with great circumspect numerically (Wen et al., 2023).

It has not been established within a rigorous framework how a posteriori error estimates and adaptivity can be developed, however, in a different way with regard to time-fractional PDEs. Unless the system defined lacks frequency memory, the local error indicators of fractional systems are difficult to compute as compared to the classical parabolic problems due to so-called effects of the temporal memory coupled admirable temporal levels. Both space and time adaptive FEM refinement will, in its turn, be in a young phase merely (Rao et al., 2023).

Potential Improvements

Several possibilities are available in the context of improved versions of solutions to TFPDEs through FEM.

Algorithms of fast time of integration Even the efficient development of the algorithms of the fractional operators is important. Following the price of no stability, methods such as driver compression (kernel compression, fast fourier convolution), truncation of the memory (memory truncation), etc., may radically reduce the number

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of calculated values (Baffet et al., 2023). Hybrid FEM-spectral schemes may also be fast in time stepping and may permit the flexibility in space.

Space time adaptation refinery In order to enhance significantly the efficiency, adaptive FEM built using a posteriori estimators relying on the goals or the residuals could be used to inform the adaptive FEM implementation (Rao et al., 2023). Managerial Refinement Managerial refinement (AMR) does allow the best spatial response and spatial resolution against fitting where needed, as at sing-reactant or sharp gradients.

High order and continuous Galerkin formulations Running the framework to hp-FEM and discontinuous Galerkin (DG) may still provide hp or superconvergent precision to smooth space-time troubles (Mustapha, 2020; He et al., 2022). It has been realized that DG formulations are particularly well suited to problems of non-smooth data and complex boundary conditions due to local conservation properties.

Nonlinear extensions Multiphysics Multiphysics Multiphysics is the ability to generalize the calculus of a system to a number of interacting physical fields. Models commonly used in the application of Fractional In the real world, application Fractional models traditionally involve nonlinear or coupled fractions, such as fractional reaction diffusion equations or fractional Navier Stokes equations. The combination of coupling between FEM and iterative nonlinear resolvers or operator-splitting methods could potentially expand possibilities of usage of the framework (Zhang et al., 2024).

Low-order modeling and artificial intelligence A physicists-informed models (PINN) as a data-zeroe surrogate based on data-zeroe surrogates (data-zeroe fatwise surrogates) could be useful in large-scale parameter identification and quantities of uncertainty in large-scale systems subject to the fraction force (Kharazmi et al., 2021).

Specialization and parallelization The access to the high-performance computing (HPC) environments could also result in the identification of real-time or massive accumulation of the 3D images in a domain area of fractional operator enhancing in a assistance of grammatisch, which actualizes its territories, by consequence of solid materials, and the procedural utilization of the computerized end, the now GPU (Huang et al., 2024).

Extensions to Higher Dimensions and Complex Physics

Extending the FEM framework to **multi-dimensional** and **multi-physics** fractional problems presents exciting research opportunities. In **3D heterogeneous domains**, FEM retains its advantage over other numerical methods due to geometric flexibility, though efficient solvers and parallel preconditioning remain essential (Li et al., 2023). Furthermore, time-fractional models can be combined with **space-fractional operators**, leading to **space–time fractional PDEs** that describe anomalous transport more comprehensively (Zhao et al., 2023). These hybrid equations demand sophisticated FEM formulations that integrate fractional Laplacian operators using nonlocal basis functions or meshless kernels.

In **multiphysics systems**, fractional operators can model memory in diffusion–reaction or viscoelastic flow problems. Coupled FEM frameworks, where fractional submodels interact with classical ones (e.g., thermal–mechanical coupling in viscoelastic solids), can be explored for predictive simulation in material science and

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biophysics (Jia et al., 2023).

Overall, the finite element framework presented for time-fractional PDEs represents a mathematically sound and practically efficient tool for simulating complex systems with memory. Its strengths lie in theoretical stability, flexibility, and proven convergence, while limitations primarily concern computational complexity and adaptivity. Future research should focus on **fast algorithms, adaptive strategies, and data-driven hybridization** to enable scalable, accurate, and real-time fractional modeling across disciplines such as geophysics, biomedicine, and materials science.

Conclusion

In this work, Caputo type of derivatives controlled TFPEs was represented by a detailed finite element representation of more generic time fractional variants of the more base deformation, TFPD, that set description of inherent dynamic memory, and non-locality of diffusion and viscoelastic events. Proposed solution Junctive spatial Galerkin discretization and quadrature-based temporal discretization schemes, which are L1 and Convolution Quadrature (CQ Convolution Quadrature BDF2) schemes, are proposed to offer an accurate and stable ND discretization with efficacies Their Sonis studies provided indications of the adequate existence and uniqueness of weak solution by the LaxMilgram lemma and provided rigorous tributes of error to L1 and between quality cost on the plain, CQBDF2, O(h2 + t 2)/L1 and CQBDF2. The prediction of this nature was actually observed to be successful through numerical experiments on benchmark problems and empirical findings showed that theoretical and experimental findings are actually very close. In addition, the structure was not strict as regards to the management of heterogeneous coefficients, complex geometries as weaker than singular temporal profiles. The method towards the problems occasioned by the non-local memory kernel and computational expense that will not contravening the strength and adaptability of the method render the approach sufficiently relevant in conditions of discovering the real-world procedures like anomalous diffusion, viscoelastic deformation, and biological transport. The additional directions of research consist in developing adaptive algorithms to enable the use of space-time, fast convolution schemes, and hybrid models, according to fast-processing on the basis of FEM as well as machine learning, to apply to more higher dimensional, non-linear, and more-term fractional systems.

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